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Hilbert-style formalism for two-dimensional notions of consequence

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Dedication

I dedicate this thesis to my parents, Adalcina and Jezuzeni, to my brother Davi, to my friends Douglas, Geovanna, Paulla and Roberto, and to my advisors, João Marcos and Sérgio Marcelino. Despite the difficulties due to the coronavirus pandemic, I never felt alone or discouraged during the development of this research, thanks to the company, guidance and support of these amazing people.

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Hilbert-style formalism for two-dimensional notions of consequence

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Abstract

The present work proposes a two-dimensional Hilbert-style deductive formalism (H-formalism) for B-consequence relations, a class of two-dimensional logics that generalize the usual (Tarskian, one-dimensional) notions of logic. We argue that the two-dimensional environment is appropriate to the study of bilateralism in logic, by allowing the primitive judgments of assertion and denial (or, as we prefer, the cognitive attitudes of acceptance and rejection) to act on independent but interacting dimensions in determining what-follows-from-what. In this perspective, our proposed formalism constitutes an inferential apparatus for reasoning over bilateralist judgments. After a thorough description of the inner workings of the proposed proof formalism, which is inspired by the one-dimensional symmetrical Hilbert-style systems, we provide a proof-search algorithm for finite analytic systems that runs in at most exponential time, in general, and in polynomial time when only rules having at most one formula in the succedent are present in the concerned system. We delve then into the area of two-dimensional non-deterministic semantics via matrix structures containing two sets of distinguished truth-values, one qualifying some truth-values as accepted and the other as rejected, constituting a semantical path for bilateralism in the two-dimensional environment. We present an algorithm for producing analytic two-dimensional Hilbert-style systems for sufficiently expressive two-dimensional matrices, as well as some streamlining procedures that allow to considerably reduce the size and complexity of the resulting calculi. For finite matrices, we should point out that the procedure results in finite systems. In the end, as a case study, we investigate the logic of formal inconsistency called **mCi** with respect to its axiomatizability in terms of Hilbert-style systems. We prove that there is no finite one-dimensional Hilbert-style axiomatization for this logic, but that it inhabits a two-dimensional consequence relation that is finitely axiomatizable by a finite two-dimensional Hilbert-style system. The existence of such system follows directly from the proposed axiomatization procedure, in view of the sufficiently expressive 5-valued non-deterministic bidimensional semantics available for the mentioned two-dimensional consequence relation.

Keywords: two-dimensional consequence relations, Hilbert-style proof systems, non-deterministic semantics, mCi

Formalismo ao estilo de Hilbert para noções de consequência bidimensionais

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Resumo

O presente trabalho propõe um formalismo dedutivo bidimensional à Hilbert (H-formalismo) para relações de B-consequência, uma classe de lógicas bidimensionais que generalizam as noções usuais (Tarskianas, unidimensionais) de lógica. Nós argumentamos que o ambiente bidimensional é apropriado para o estudo do bilateralismo em lógica, por permitir que julgamentos primitivos de asserção e denegação (ou, como preferimos, as atitudes cognitivas de aceitação e rejeição) ajam em dimensões independentes e capazes de interagir entre si ao determinar as inferências válidas de uma lógica. Nessa perspectiva, o formalismo proposto constitui um aparato inferencial para raciocinar sobre julgamentos bilateralistas. Após uma descrição detalhada do funcionamento do formalismo proposto, o qual é inspirado nos sistemas de Hilbert simétricos, nós provemos um algoritmo de busca de demonstrações que executa em tempo exponencial, em geral, e em tempo polinomial quando apenas regras contendo no máximo uma fórmula no sucedente estão presentes no sistema em questão. Então, nós passamos a investigar semânticas não-determinísticas bidimensionais por meio de estruturas de matrizes contendo dois conjuntos de valores distinguidos, um qualificando alguns valores de verdade como aceitos, e o outro, alguns valores como rejeitados, constituindo um caminho semântico para o bilateralismo no ambiente bidimensional. Nós apresentamos também um algoritmo para a produção de sistemas de Hilbert bidimensionais para matrizes não-determinísticas bidimensionais suficientemente expressivas, bem como alguns procedimentos de simplificação que permitem reduzir consideravelmente o tamanho e a complexidade do sistema resultante. Para matrizes finitas, vale apontar, o procedimento resulta em sistemas finitos. Ao final, como estudo de caso, investigamos a lógica da inconsistência formal chamada **mCi** quanto à sua axiomatizabilidade por sistemas ao estilo de Hilbert. Demonstramos que não há sistemas de Hilbert finitos unidimensionais que capturem essa lógica, mas que ela habita uma relação de consequência bidimensional finitamente axiomatizável por um sistema de Hilbert bidimensional. A existência desse sistema segue diretamente do algoritmo de axiomatização proposto, em vista da semântica bidimensional 5-valorada não-determinística suficientemente expressiva que determina a relação de consequência bidimensional mencionada.

Palavras-chave: relações de consequência bidimensionais, sistemas de demonstração ao estilo de Hilbert, semânticas não-determinísticas, mCi.

Contents

1. Introduction

A logical system is commonly understood, nowadays, to presuppose a *formal language* equipped with a notion of *consequence* that connects expressions of the said language and satisfies certain closure properties. The formal languages used in doing logic are typically generated in a purely syntactical fashion and induce a collection of formulas with an algebraic structure. The associated notion of consequence may be obtained in a number of ways, in particular, proof-theoretically, through different kinds of *proof formalisms*, or semantically, through the use of a *satisfaction relation* defined with the help of some class of mathematical objects. In all cases, the very structure of the formulas is usually productively explored when a given specific consequence relation involving them is defined. At their simplest, consequence relations end up consisting in collections of the so-called *consecutions*, that is, statements involving formulas or certain collections thereof.

In the realm of Proof Theory, each choice of proof formalism gives rise to a different class of deductive systems. In what follows, we shall call *sequents* the syntactical objects manipulated by a given deductive system, and we will let *G-systems* stand for "Gentzen-style systems" and *H-systems* stand for "Hilbert-style systems", two popular proof formalisms. The sequents manipulated by H-systems, on the one hand, are precisely the components of consecutions, namely their antecedents or their succedents. The sequents manipulated by G-systems, on the other hand, have both an antecedent and a succedent, and the rules of those systems also involve contexts, or side formulas. While such contexts allow for hypothetical judgments to be formulated, their absence, in H-systems, has the interesting effect of forcing the deductive system to provide evidence for judgments that have the exact form of the consecutions under scrutiny, free of metalinguistic machinery. A further bonus feature worth mentioning is that H-systems allow for a clean and straightforward notion of combination of logics: the merging of two H-systems induces the smallest consequence relation in the joint language containing the consequence relations induced by each one of them in separate.

Despite all that, traditional H-systems — those whose rules of inference have sets of formulas as antecedents and single formulas as succedents — are considered hard to work with due to the lack of control over of the search space when producing derivations, which is usually guaranteed in other formalisms (G-systems, for example) via analyticity results. Recently, C. Caleiro and S. Marcelino [43, 17] showed that, by adopting the Hilbert-style formalism introduced by D. Shoesmith and T. Smiley [57], which slightly generalizes the traditional H-systems to allow for sets of formulas also in the succedents of rules, one can produce analytic H-systems for a very representative class of logics determined by a class of non-deterministic semantical structures. Since then, Hilbert-style systems are no longer only interesting from a theoretical point of view, but also from the practical perspectives of automatic system generation, proof search and automated reasoning.

When defining a notion of consequence, one has usually in mind a certain kind of judgment that somehow governs what-follows-from-what in the reasoning underlying the corresponding logical system. In contemporary logic there has been a persistent bias towards a specific kind of judgment identified with the speech act of *assertion*. Indeed, *unilateralist* consequence relations have focused exclusively on assertion, and have insisted in reducing its polar opposite, *denial*, to the assertion of a negation.

In contrast, one may adopt *bilateralism*, an approach to logic in which denial is treated as a primitive judgment, on a par with assertion. The first ideas on bilateralism date back to the seventies, when K. Bendall [10] discussed the philosophical advantages for the study of the meaning of negation in not employing a negation connective to represent the acts of negative judgment, disbelief and denial. In 1983, H. Price [51] defended the view of taking denial conditions together with assertion conditions into consideration in determining the sense of propositions. Some years later, the same author took those ideas to philosophical investigations about negation [52]. In [59], T. Smiley introduced rules of rejection and presented bilateralist axiomatizations as a path to categoricity, that is, the exclusion of unintended models by means of rules of inference. In 2000, I. Rumfitt [54] and L. Humberstone [37] presented signed natural deduction systems for intuitionistic and classical logics and discussed the applicability of bilateralism in the study of the meaning of logical constants via deductive systems, as well as in the differentiation between intuitionistic and classical reasonings. Since then, despite the criticisms and debates regarding these ideas [30, 56, 25, 55, 31], bilateralism has been present in many lines of research. Just to name a few, it appears in works on philosophical proof theory of classical and intuitionistic logics [40, 39], in proof-theoretical investigations of bi-intuitionistic logic [22] and in bilateral generalizations of Tarskian consequence relations [23, 13].

One may be tempted to implement bilateralism by just employing Gentzen-style systems that manipulate symmetrical sequents, that is, sequents consisting of a set of formulas in the antecedent and also a set of formulas in the succedent — as opposed to asymmetrical sequents, which only allow for a single formula in the succedent. This symmetry allows not only to stipulate that some formula must be denied when a given set of formulas is asserted, but also that some formula must be asserted when a given set of formulas is denied. What may bring some embarrassment to this approach is the fact that denial is still taken at large to be equivalent to non-assertion (and assertion is taken to be equivalent to non-denial). Moreover, in the endgame, the associated notion of consequence remains unaffected, governed by a single judgment; in other words, still a unilateralist consequence.

One way of letting these two kinds of judgments coexist without necessarily allowing them to interfere with one another consists in attaching to the underlying formulas a force indicator or signal, say $+$ for assertion and $-$ for denial [54, 23]. For example, the consecution $-(\varphi \to \psi) \vdash +\varphi$ describes a rule in the bilateral axiomatization of classical logic given in [54], representing the impossibility of, at once, denying $\varphi \to \psi$ while failing to assert φ. In [13], a concurrent approach of working with a two-dimensional notion of consequence is offered, allowing for the cognitive attitudes of acceptance and rejection to act over two separate logical dimensions and taking their interaction into consideration in determining the meaning of logical connectives and of the statements involving them. The aforementioned inference, for instance, would be expressed by the two-dimensional judgment $\frac{\varnothing}{\varnothing}|\frac{\varphi}{\varphi\to\psi}$, which is intended to enforce that an agent is not expected to find reasons for rejecting $\varphi \to \psi$ while failing to find reasons for accepting φ. More generally, where $\Phi_{\textsf{Y}}, \Phi_{\textsf{⅄}}, \Phi_{\textsf{N}}$ and $\Phi_{\textsf{И}}$ are sets of formulas, a judgment $\frac{\Phi_{\textsf{И}}}{\Phi_{\textsf{Y}}}|\frac{\Phi_{\textsf{⅄}}}{\Phi_{\textsf{N}}}$ is intended to enforce that an agent is not expected to find reasons for simultaneously accepting the formulas in $\Phi_{\textsf{Y}}$, non-accepting the formulas in $\Phi_{\textsf{⅄}}$, rejecting the formulas in $\Phi_{\textsf{N}}$ and non-rejecting the formulas in $\Phi_{\textsf{И}}$. If, in particular, $\Phi_{\textsf{Y}}$ is empty, the judgment imposes no commitment with respect to the acceptance of any formula. This holds similarly for the sets $\Phi_{\textsf{⅄}}$, $\Phi_{\textsf{N}}$ and $\Phi_{\textsf{И}}$.

From a semantical standpoint, two-dimensional consequences may be actualized by the canonical notion of entailment induced by a so-called *nd*-B-*matrix* [13], a partial non-deterministic logical matrix in which the latter judgments, or cognitive attitudes, are represented by separate collections of truth-values. Non-deterministic logical matrices have been extensively investigated in recent years, and proved useful in the construction of effective semantics for many families of logics in a systematic and modular way [6, 45, 19, 18]. These structures interpret the logical constants as mappings allowed to output nonempty sets of values — in contrast to the notion of (deterministic) logical matrices that traditionally appears in the study of many-valued logics, in which the mappings may only output a single value. In this work, as in [7], we consider more general structures called *partial* non-deterministic logical matrices, in which the empty set is also allowed as output of the interpretations.

From the proof-theoretical perspective, C. Blasio [11] introduced G-systems that manipulate two-dimensional sequents. A Hilbert-style proof formalism for this two-dimensional notion of logic, however, is currently missing. One of the goals of the present thesis is to fill this gap. Our approach consists in a generalization of the Hilbert-style formalism of D. Shoesmith and T. Smiley [57] by allowing the rules of inference to deal with pairs of sets of formulas, and letting derivations be trees whose nodes come labelled with such pairs and result from expansions determined by the rules. The first component of these pairs is intended to represent the accepted formulas, while the second component represents the rejected formulas.

More than just introducing two-dimensional H-systems, we will also generalize the works of C. Caleiro and S. Marcelino [43, 17] with respect to the extraction of analytic axiomatizations from logical matrices satisfying a property of sufficient expressiveness. When a matrix is sufficient expressive, each truth-value is characterized by a collection of formulas over a single propositional variable. Our extension of such property to nd-B-matrices will show that adding a new dimension (represented by a new set of distinguished values) to the matrix structure potentially increases the expressiveness of the linguistic resources of the associated logic, in the sense that with the same language we may characterize more values than we could with a single dimension. Besides being a useful result on axiomatizability, our generalization also facilitates the study of bilateralism from the perspective of Hilbert-style systems and many-valued logics.

As already mentioned, the analyticity of Hilbert-style systems is extremely important for their usability. In fact, this property allows for bounded proof search and countermodel search, and for the design of a simple recursive decision algorithm [43] that runs in exponential time in general, and in polynomial time when the rules of inference have at most one formula in the succedent. Both algorithms — the system generation and the proof search — are studied and implemented using the `C++` programming language in the present work, for both the one-dimensional and the two-dimensional cases.

Finally, we will show how to take advantage of the additional expressiveness power of two-dimensional matrices in order to produce a finite and analytic two-dimensional Hilbert-style axiomatization for the one-dimensional logic of formal inconsistency called **mCi**, introduced and studied by J. Marcos in [47]. Our result contrasts with the fact — which we also prove here — that this logic is not finitely axiomatizable by one-dimensional Hilbert-style systems. What we provide is, actually, a novel way of combining two logical matrices into a single nd-B-matrix that is potentially more expressive than the ingredients of the combination.

The present document is organized as follows: Chapter 2 introduces the basic concepts and terminology involved in one-dimensional and two-dimensional notions of consequence and deductive formalisms. Chapter 3 provides a general treatment of symmetrical Hilbert-style systems and illustrates it with the one-dimensional symmetrical Hilbert-style formalism. Chapter 4 describes our proposed two-dimensional proof formalism and a proof-search and countermodel-search algorithm for analytic two-dimensional systems, proving its correctness and investigating its worst-case exponential asymptotic complexity. Chapter 5 presents the general axiomatization procedure for finite sufficiently expressive matrices, illustrating it and highlighting its modularity via the correspondence between refining a matrix and adding rules to a sound symmetrical two-dimensional

H-system. Chapter 6 shows that the logic **mCi** is not finitely axiomatizable by a one-dimensional H-system and presents a two-dimensional system for it as a product of the algorithm described in the previous chapter. In the final remarks, we reflect upon the obtained results and indicate some directions for future developments. Finally, Appendix A provides instructions on how to execute the `C++` implementation of the axiomatization and proof-search algorithms.

2. Theoretical background

2.1. Algebras and languages

Given a mapping $f : X \to Y$, we denote its domain X by $\mathsf{dom}(f)$, its codomain Y by $\mathsf{codom}(f)$ and its range $\{f(x) \mid x \in X\} \subseteq Y$ by $\mathsf{ran}(f)$. In case $Y \subseteq Z$, we let $f{\uparrow}Z : X \to Z$ be the mapping such that $f{\uparrow}Z(x) = f(x)$ for all $x \in Y$. The *restriction of f to $W \subseteq X$* is the function $g : W \to Y$ such that $g(x) := f(x)$ for every $x \in W$. The power set of a set X is denoted by $\mathsf{Pow}(X)$. We write the natural extension of f to the domain $\mathsf{Pow}(X)$ using the same symbol f, but, given $Z \subseteq X$, we write $f[Z]$ for the image of Z — the set $\{f(x) \mid x \in Z\}$ — under f. A *multifunction* on Y is a mapping $f : X \to \mathsf{Pow}(Y)$, seen as providing *output alternatives* (instead of a single element) for each element in $\mathsf{dom}(f)$. The cardinality of a set X is denoted by $|X|$. The set of natural numbers is denoted by ω.

A *propositional signature* is a family $\Sigma := \{\Sigma_k\}_{k\in\omega}$, where each Σ_k is a collection of k-ary *connectives*. In case Σ_k is finite for every $k \in \omega$, we say that Σ is *finite*. A *non-deterministic algebra over* Σ, or simply Σ-*nd-algebra*, is a structure $\mathbf{A} := \langle A, \cdot_{\mathbf{A}} \rangle$, such that A is a non-empty collection of elements called the *carrier* of $\mathbf{A}$, and, for each $k \in \omega$ and $© \in \Sigma_k$, the multifunction $©_{\mathbf{A}} : A^k \to \mathsf{Pow}(A)$ is the *interpretation of © in* $\mathbf{A}$. When Σ and A are finite, we say that $\mathbf{A}$ is *finite*. When the range of all interpretations of $\mathbf{A}$ contains only singletons, $\mathbf{A}$ is said to be a *deterministic algebra over* Σ, or simply a Σ-*algebra*, meeting the usual definition from Universal Algebra [16].

Notice that Σ-nd-algebras allow for *partial* interpretations of connectives, that is, we may have $\varnothing \in \mathsf{ran}(©_{\mathbf{A}})$ for some $© \in \Sigma_k$, a possibility that is often taken out in many treatments of non-deterministic algebraic structures — usually called 'hyperalgebras' or 'multialgebras' [32]. The Σ-nd-algebras with no partial interpretations are said to be *total.*

Example 1. *Let $\mathcal{V}_4 := \{\mathbf{f}, \bot, \top, \mathbf{t}\}$ and consider a signature Σ^{FDE} containing but two binary connectives, $\wedge$ and $\vee$, and one unary connective, $\neg$. Next, define the Σ^{FDE}-nd-algebra $\mathbf{I} := \langle \mathcal{V}_4, \cdot_{\mathbf{I}} \rangle$ that interprets these connectives according to the following (non-deterministic) truth-tables (here and below, braces will be omitted in the images of the interpretations):*

$\wedge_{\mathbf{I}}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\bot$	$\mathbf{f}$	$\mathbf{f}, \bot$	$\mathbf{f}$	$\mathbf{f}, \bot$
$\top$	$\mathbf{f}$	$\mathbf{f}$	$\top$	$\top$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}, \bot$	$\top$	$\mathbf{t}, \top$

$\vee_{\mathbf{I}}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}, \top$	$\mathbf{t}, \bot$	$\top$	$\mathbf{t}$
$\bot$	$\mathbf{t}, \bot$	$\mathbf{t}, \bot$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\top$	$\mathbf{t}$	$\top$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

	$\neg_{\mathbf{I}}$
$\mathbf{f}$	$\mathbf{t}$
$\bot$	$\bot$
$\top$	$\top$
$\mathbf{t}$	$\mathbf{f}$

The non-deterministic character of this algebra can be observed in several entries of the interpretations of $\wedge$ and $\vee$. For example, the output of $\wedge_{\mathbf{I}}$ when taking the input $(\bot, \bot)$ is the set $\{\mathbf{f}, \bot\}$, meaning that there are two possible values or choices provided by this interpretation under this input.

For the remainder of this subsection, let $\mathbf{A}$ and $\mathbf{B}$ be arbitrary Σ-nd-algebras. We say that $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$, denoted by $\mathbf{B} \leqslant \mathbf{A}$, when $B \subseteq A$ and, for all $k \in \omega$, $© \in \Sigma_k$ and $y_1, \ldots, y_k \in B$, we have $©_{\mathbf{B}}(y_1, \ldots, y_k) \subseteq ©_{\mathbf{A}}(y_1, \ldots, y_k)$. Every $X \subseteq A$ induces a subalgebra $\mathbf{A}_X := \langle X, \cdot_{\mathbf{A}_X} \rangle$ of $\mathbf{A}$, with $©_{\mathbf{A}_X}(x_1, \ldots, x_k) := ©_{\mathbf{A}}(x_1, \ldots, x_k) \cap X$, for all $k \in \omega$, $© \in \Sigma_k$ and $x_1, \ldots, x_k \in X$. The collection of all subsets of A whose elements are in at least one induced total subalgebra of $\mathbf{A}$ is denoted by $\mathbb{T}(\mathbf{A})$, that is

$$\mathbb{T}(\mathbf{A}) := \bigcup_{\substack{\varnothing \neq X \subseteq A \\ \mathbf{A}_X \text{ is total}}} \mathsf{Pow}(X). \tag{2.1}$$

Example 2. *Let* $\mathbf{E} := \langle \mathcal{V}_4, \cdot_{\mathbf{E}} \rangle$ *be the* Σ^{FDE}*-nd-algebra whose interpretations are given by the following tables:*

$$\begin{array}{c|cccc}
\wedge_{\mathbf{E}} & \mathbf{f} & \bot & \top & \mathbf{t} \\ \hline
\mathbf{f} & \mathbf{f} & \mathbf{f} & \mathbf{f} & \mathbf{f} \\
\bot & \mathbf{f} & \bot & \mathbf{f} & \bot \\
\top & \mathbf{f} & \mathbf{f} & \top & \top \\
\mathbf{t} & \mathbf{f} & \bot & \top & \mathbf{t}
\end{array}
\qquad
\begin{array}{c|cccc}
\vee_{\mathbf{E}} & \mathbf{f} & \bot & \top & \mathbf{t} \\ \hline
\mathbf{f} & \mathbf{f} & \bot & \top & \mathbf{t} \\
\bot & \bot & \bot & \mathbf{t} & \mathbf{t} \\
\top & \top & \mathbf{t} & \top & \mathbf{t} \\
\mathbf{t} & \mathbf{t} & \mathbf{t} & \mathbf{t} & \mathbf{t}
\end{array}
\qquad
\begin{array}{c|c}
 & \neg_{\mathbf{E}} \\ \hline
\mathbf{f} & \mathbf{t} \\
\bot & \bot \\
\top & \top \\
\mathbf{t} & \mathbf{f}
\end{array}$$

It is easy to check that $\mathbf{E}$ *is a subalgebra of the* Σ*-nd-algebra* $\mathbf{I}$ *given in the previous example.*

Example 3. *Let* $\mathbf{K} := \langle \mathcal{V}_4, \cdot_{\mathbf{K}} \rangle$ *be the* Σ^{FDE}*-nd-algebra such that* $\cdot_{\mathbf{K}}$ *is the same as* $\cdot_{\mathbf{E}}$ *except that* $\wedge_{\mathbf{K}}(\top, \bot) = \vee_{\mathbf{K}}(\top, \bot) = \vee_{\mathbf{K}}(\bot, \top) = \wedge_{\mathbf{K}}(\bot, \top) = \varnothing$*, as the following tables show:*

$$\begin{array}{c|cccc}
\wedge_{\mathbf{K}} & \mathbf{f} & \bot & \top & \mathbf{t} \\ \hline
\mathbf{f} & \mathbf{f} & \mathbf{f} & \mathbf{f} & \mathbf{f} \\
\bot & \mathbf{f} & \bot & \varnothing & \bot \\
\top & \mathbf{f} & \varnothing & \top & \top \\
\mathbf{t} & \mathbf{f} & \bot & \top & \mathbf{t}
\end{array}
\qquad
\begin{array}{c|cccc}
\vee_{\mathbf{K}} & \mathbf{f} & \bot & \top & \mathbf{t} \\ \hline
\mathbf{f} & \mathbf{f} & \bot & \top & \mathbf{t} \\
\bot & \bot & \bot & \varnothing & \mathbf{t} \\
\top & \top & \varnothing & \top & \mathbf{t} \\
\mathbf{t} & \mathbf{t} & \mathbf{t} & \mathbf{t} & \mathbf{t}
\end{array}
\qquad
\begin{array}{c|c}
 & \neg_{\mathbf{K}} \\ \hline
\mathbf{f} & \mathbf{t} \\
\bot & \bot \\
\top & \top \\
\mathbf{t} & \mathbf{f}
\end{array}$$

Observe that this is an example of a non-deterministic algebra with partial interpretations, and note that $\mathbb{T}(\mathbf{K}) = \{X \subseteq \mathcal{V}_4 \mid \{\top, \bot\} \not\subseteq X\}$.

A mapping $v : A \to B$ is a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ when, for all $k \in \omega$, $© \in \Sigma_k$ and $x_1, \ldots, x_k \in A$, we have

$$f[©_{\mathbf{A}}(x_1, \ldots, x_k)] \subseteq ©_{\mathbf{B}}(f(x_1), \ldots, f(x_k)). \tag{2.2}$$

Notice that, in case $\mathbf{A}$ is deterministic (provided we identify singletons with their elements) we may simply write '$\in$' in the place of '$\subseteq$', and when both $\mathbf{A}$ and $\mathbf{B}$ are deterministic, we may use '$=$' instead, matching thus the usual notion of homomorphism for Σ-algebras [16]. The set of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}$ is denoted by $\mathsf{Hom}(\mathbf{A}, \mathbf{B})$. When $\mathbf{A} = \mathbf{B}$, we write $\mathsf{End}(\mathbf{A})$ for the set of *endomorphisms on* $\mathbf{A}$.

Let P be a denumerable set of *propositional variables* and Σ be a propositional signature. The absolutely free (deterministic) Σ-algebra freely generated by P is denoted by $\mathbf{L}_\Sigma(P)$ and called the Σ*-language generated by* P. The elements of $L_\Sigma(P)$ are called Σ*-formulas*, and those among them that are not propositional variables are called Σ*-compounds*. Given $\Phi \subseteq L_\Sigma(P)$, we denote by Φ^{c} the set $L_\Sigma(P) \backslash \Phi$. When speaking informally, we may refer to Σ-formulas simply as *sentences* or *propositions*. The homomorphisms from $\mathbf{L}_\Sigma(P)$ to $\mathbf{A}$ are called *valuations on* $\mathbf{A}$, and we let $\mathsf{Val}(\mathbf{A}) := \mathsf{Hom}(\mathbf{L}_\Sigma(P), \mathbf{A})$. Additionally, endomorphisms on $\mathbf{L}_\Sigma(P)$ are dubbed *substitutions*, and we let $\mathsf{Subs}^P_\Sigma := \mathsf{End}(\mathbf{L}_\Sigma(P))$. When there is no risk of confusion, we may omit the set of propositional variables and simply write Subs_Σ.

Lemma 4. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* Σ*-nd-algebras such that* $\mathbf{B} \leqslant \mathbf{A}$*. Then* $\{v{\uparrow}A \mid v \in \mathsf{Val}(\mathbf{B})\} \subseteq \mathsf{Val}(\mathbf{A})$*.*

Proof. See [7], Proposition 4.8. $\square$

Given $\varphi \in L_\Sigma(P)$, let $\mathsf{subf}(\varphi)$ be the set of *subformulas of* φ — that is, $\mathsf{subf}(\varphi) := \{p\}$ if $\varphi = p \in P$ and $\mathsf{subf}(©(\psi_1, \ldots, \psi_k)) := \{\varphi\} \cup \bigcup_{i=1}^k \mathsf{subf}(\psi_i)$ if $\varphi = ©(\psi_1, \ldots, \psi_k)$ — and let $\mathsf{props}(\varphi)$ denote the set of propositional variables occurring in φ — that is, $\mathsf{props}(\varphi) := \mathsf{subf}(\varphi) \cap P$. If $\mathsf{props}(\varphi) = \{p_1, \ldots, p_k\}$, we say that φ is k-ary and we let $\varphi_{\mathbf{A}} : A^k \to \mathsf{Pow}(A)$ be *the* k*-ary multifunction on* $\mathbf{A}$ *induced by* φ, where, for all $x_1, \ldots, x_k \in A$, we have $\varphi_{\mathbf{A}}(x_1, \ldots, x_k) = \{v(\varphi) \mid v \in \mathsf{Val}(\mathbf{A}) \text{ and } v(p_i) = x_i, for 1 \leq i \leq k\}$. Moreover, given $\psi_1, \ldots, \psi_k \in L_\Sigma(P)$, we write $\varphi(\psi_1, \ldots, \psi_k)$ for the Σ-formula $\varphi_{\mathbf{L}_\Sigma(P)}(\psi_1, \ldots, \psi_k)$, and, in case $\Phi \subseteq L_\Sigma(P)$ is a set of k-ary Σ-formulas, we let

$\Phi(\psi_1, \ldots, \psi_k) := \{\varphi(\psi_1, \ldots, \psi_k) \mid \varphi \in \Phi\}$. Another function on $L_\Sigma(P)$ gives us the *size* of a Σ-formula as the cardinality of the multiset of subformulas (that is, when repetition is allowed), being defined as $\mathsf{size}(\varphi) := 1$, if $\varphi \in P$, and $\mathsf{size}(©(\psi_1, \ldots, \psi_k)) := 1+\sum_{i=1}^{k} \mathsf{size}(\psi_i)$ otherwise.

We call *value-assignments on* $\mathbf{A}$ the mappings $f : \Phi \to A$, where $\Phi \subseteq L_\Sigma(P)$. A value-assignment f on $\mathbf{A}$ is called *legal* when the following conditions are satisfied:

(**L1**) Φ is closed under subformulas;

(**L2**) $f(©(\varphi_1, \ldots, \varphi_k)) \in ©_{\mathbf{A}}(f(\varphi_1), \ldots, f(\varphi_k))$, for all $©(\varphi_1, \ldots, \varphi_k) \in \Phi$;

(**L3**) $\mathsf{ran}(f) \in \mathbb{T}(\mathbf{A})$.

A fundamental property of $\mathbf{L}_\Sigma(P)$ is the *universal mapping property*, establishing that, when $\mathbf{A}$ is deterministic, every value-assignment $f : P \to A$ (notice that the domain here is P) can be uniquely extended to a homomorphism from $\mathbf{L}_\Sigma(P)$ to $\mathbf{A}$. For Σ-nd-algebras in general, we have the result presented in Theorem 5 below, which is sometimes referred to as *effectiveness* or *analyticity* [2] and is responsible for many of the useful features of non-deterministic semantics. It was first proved for the total non-deterministic case, and then extended to the partial case in [7], both in the context of (partial non-deterministic) logical matrices. Condition (**L3**) is what allowed for the extension of this result. We reproduce here the proof for self-containment.

Theorem 5. *Let* $\mathbf{A}$ *be a* Σ*-nd-algebra. Then every legal value-assignment on* $\mathbf{A}$ *can be extended to a valuation on* $\mathbf{A}$.

Proof. Let $f : \Phi \to A$ be a legal value-assignment on a Σ-nd-algebra $\mathbf{A}$. Then, by (**L3**), $f[\Phi] \in \mathbb{T}(\mathbf{A})$, meaning that $f[\Phi] \subseteq X$, for some nonempty $X \subseteq A$, such that $\mathbf{A}_X$ is a total Σ-nd-algebra. We proceed to define a valuation f^* on $\mathbf{A}_X$ extending f, which, in view of Lemma 4, will give us the desired result. In this direction, recursively define a mapping $f^* : L_\Sigma(P) \to X$ by setting (a): $f^*(\varphi) := f(\varphi)$ for each $\varphi \in \Phi$, (b): $f^*(p) := y$, for each $p \in P \backslash \Phi$ and a fixed $x \in X$, and (c): for all $©(\psi_1, \ldots, \psi_k) \notin \Phi$, $f^*(©(\psi_1, \ldots, \psi_k)) := y$, for a choice of $y \in ©_{\mathbf{A}_X}(f^*(\psi_1), \ldots, f^*(\psi_k))$ — always available, as $\mathbf{A}_X$ is total. Clearly, (a), (b) and (c) guarantee that f^* is well-defined and extends f. In order to prove that f^* is a homomorphism, let $\varphi := ©(\psi_1, \ldots, \psi_k) \in L_\Sigma(P)$. If, on the one hand, $\varphi \in \Phi$, we have $\psi_1, \ldots, \psi_k \in \Phi$, as Φ is closed under subformulas, by **(L1)**; then, by (a), $f^*(\psi_i) = f(\psi_i)$, for all $1 \leq i \leq k$, and, by (a) again and **(L2)**, $f^*(©(\psi_1, \ldots, \psi_k)) = f(©(\psi_1, \ldots, \psi_k)) \in ©_{\mathbf{A}}(f(\psi_1), \ldots, f(\psi_k)) \cap X = ©_{\mathbf{A}_X}(f(\psi_1), \ldots, f(\psi_k)) = ©_{\mathbf{A}_X}(f^*(\psi_1), \ldots, f^*(\psi_k))$. If, on the other hand, $\varphi \notin \Phi$, (c) guarantees the desired result. $\square$

2.2. A broad account of logic

The present work is not committed to a specific notion of logic. Actually, we will deal with and compare a variety of notions, some of them well-known, like those consisting in relations over formulas satisfying the Tarskian axioms, and some others probably new, at the present moment, to most readers. For this reason, the word "logic" will seldomly appear without qualifiers. As may happen sometimes in informal prose, however, we now give a very broad account of logic, to be considered when this word appears alone in a sentence.

For us, a *logic (over Σ)* is a subset of the set of *statements* determined by a chosen *logical framework*. The frameworks of interest in this work are the following:

- Set-Fmla: logics are 2-place relations $\vdash \subseteq \mathsf{Pow}(L_\Sigma(P)) \times L_\Sigma(P)$, that is, statements have the form of pairs (Φ, ψ), which we denote by $\Phi \succ \psi$, for Φ a set of formulas and ψ a formula.
- Set-Set: logics are 2-place relations $\rhd \subseteq \mathsf{Pow}(L_\Sigma(P)) \times \mathsf{Pow}(L_\Sigma(P))$, that is, statements have the form of pairs (Φ, Ψ), being denoted by $\Phi \succ \Psi$, for Φ and Ψ sets of formulas. Observe that every Set-Set logic $\rhd$ induces a Set-Fmla logic $\vdash_{\rhd}$, called *the* Set-Fmla *companion of* $\rhd$, such that, for all $\Phi \cup \{\psi\} \subseteq L_\Sigma(P)$, $\Phi \vdash_{\rhd} \psi$ if, and only if, $\Phi \rhd \{\psi\}$. In some situations, knowing that a Set-Fmla logic is the

Set-Fmla companion of a Set-Set logic enables us to draw useful conclusions about each of them (see [57]). Furthermore, we will denote by $\blacktriangleright$ the complement of $\rhd$, sometimes referred to as the *compatibility relation associated with* $\rhd$.

- Set^2-Set^2: logics are 2×2-place relations $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot} \subseteq (\mathsf{Pow}(L_\Sigma(P)))^2 \times (\mathsf{Pow}(L_\Sigma(P)))^2$, that is, statements have the form of 2×2-place tuples $((\Phi_{11}, \Phi_{21}), (\Phi_{12}, \Phi_{22}))$, which we denote by $\left(\begin{smallmatrix}\Phi_{22} & \Phi_{12}\\ \Phi_{11} & \Phi_{21}\end{smallmatrix}\right)$. A Set^2-Set^2 *consequence judgment* is an assertion of the form $\left(\begin{smallmatrix}\Phi_{22} & \Phi_{12}\\ \Phi_{11} & \Phi_{21}\end{smallmatrix}\right) \in \frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}$, and, when it holds, we write $\frac{\Phi_{22}}{\Phi_{11}}|\frac{\Phi_{12}}{\Phi_{21}}$. We will denote by $\frac{\cdot}{\cdot}\divideontimes\frac{\cdot}{\cdot}$ the complement of $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}$, sometimes called the *compatibility relation associated with* $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}$. In Section 2.4, we will see that this framework provides a rich environment for developing logics involving independent notions of acceptance and rejection. In addition, a Set^2-Set^2 logic has many aspects, which can be studied as Set-Set logics by themselves, as we shall see.

Statements belonging to a logic are, more generally, called *consecutions* of that logic. The consecutions of a logic are intended to represent the fact that, according to that logic, one or more formulas follow from, or are consequences of, other formulas. When writing sets of formulas in the scope of a statement, we usually omit curly braces. Also, given variables Φ and Ψ for sets of formulas, we write Φ, Ψ instead of $\Phi \cup \Psi$.

The frameworks Set-Fmla and Set-Set are *one-dimensional frameworks*, and logics conforming to them are called *one-dimensional logics*. On the other hand, the framework Set^2-Set^2 is a *two-dimensional framework*, and logics conforming to them are referred to as *two-dimensional logics*. In the subsections that follow, we will present and discuss some topics related to specific notions of logic, which are, in essence, classes of logics of the frameworks listed above, determined by specific properties.

2.3. One-dimensional logics

In this section, we will present different abstract definitions of one-dimensional logics considered in the literature, as well as some ways by which we may obtain concretizations of those notions via semantical structures. We will begin with the most common notions in SET-FMLA and SET-SET, the so-called consequence relations, then we will proceed to other notions motivated by some limitations of consequence relations with respect to inferential many-valuedness [42, 13].

2.3.1. Consequence relations

A SET-FMLA *consequence relation (over Σ)* is a SET-FMLA logic $\vdash^{\mathsf{t}}$ satisfying the following properties, for all $\Phi, \Psi, \{\psi\} \subseteq L_\Sigma(P)$:

(R) $\Phi, \psi \vdash^{\mathsf{t}} \psi$

(M) if $\Phi \vdash^{\mathsf{t}} \psi$, then $\Phi, \Psi \vdash^{\mathsf{t}} \psi$

(T$^+$) if $\Phi, \Psi \vdash^{\mathsf{t}} \psi$ and $\Phi \vdash^{\mathsf{t}} \varphi$ for all $\varphi \in \Psi$, then $\Phi \vdash^{\mathsf{t}} \psi$.

Properties **(R)**, **(M)** and **(T$^+$)** are called, respectively, *reflexivity*, *monotonicity* and *transitivity*. When $\vdash^{\mathsf{t}}$ satisfies, in addition, the property of *substitution-invariance*, given by

(S) if $\Phi \vdash^{\mathsf{t}} \psi$ and $\sigma \in \mathsf{Subs}_\Sigma$, then $\sigma[\Phi] \vdash^{\mathsf{t}} \sigma(\psi)$.

it is said to be *substitution-invariant*. Moreover, $\vdash^{\mathsf{t}}$ may be *finitary*, which is to say that it satisfies

(F) if $\Phi \vdash^{\mathsf{t}} \psi$, then, for some finite $\Phi^{\mathsf{f}} \subseteq \Phi$, $\Phi^{\mathsf{f}} \vdash^{\mathsf{t}} \psi$.

In case $\vdash^{\mathsf{t}}$ is a finitary consequence relation, **(T$^+$)** may be replaced by

(T) if $\Phi, \varphi \vdash^{\mathsf{t}} \psi$ and $\Phi \vdash^{\mathsf{t}} \varphi$, then $\Phi \vdash^{\mathsf{t}} \psi$.

Substitution-invariant and finitary SET-FMLA consequence relations are called *standard*.

A SET-SET *consequence relation (over Σ)* is a SET-SET logic $\rhd^{\mathsf{t}} \subseteq \mathsf{Pow}(L_\Sigma(P)) \times$

$\mathsf{Pow}(L_\Sigma(P))$ satisfying, for all $\Phi, \Psi, \Phi', \Psi' \subseteq L_\Sigma(P)$,

(O) if $\Phi \cap \Psi \neq \varnothing$, then $\Phi \rhd^{\mathsf{t}} \Psi$

(D) if $\Phi \rhd^{\mathsf{t}} \Psi$, then $\Phi, \Phi' \rhd^{\mathsf{t}} \Psi, \Psi'$

(C) if $\Pi, \Phi \rhd^{\mathsf{t}} \Psi, \Pi^{\mathsf{c}}$ for all $\Pi \subseteq L_\Sigma(P)$, then $\Phi \rhd^{\mathsf{t}} \Psi$.

Properties **(O)**, **(D)** and **(C)** are called *overlap*, *dilution* and *cut*, respectively. The relation $\rhd^{\mathsf{t}}$ is *substitution-invariant* when it satisfies

($\mathbf{S_S}$) if $\Phi \rhd^{\mathsf{t}} \Psi$ and $\sigma \in \mathsf{Subs}_\Sigma$, then $\sigma[\Phi] \rhd^{\mathsf{t}} \sigma[\Psi]$.

and it is *finitary* when it satisfies

($\mathbf{F_S}$) if $\Phi \rhd^{\mathsf{t}} \Psi$, then $\Phi^{\mathsf{f}} \rhd^{\mathsf{t}} \Psi^{\mathsf{f}}$ for some finite $\Phi^{\mathsf{f}} \subseteq \Phi$ and $\Psi^{\mathsf{f}} \subseteq \Psi$.

It is worth pointing out that, when we constrain the succedents of Set-Set consecutions to singletons, we obtain that Set-Fmla consequence relations are particular cases of Set-Set consequence relations. We may refer to consequence relations, either Set-Fmla or Set-Set, as *Tarskian logics*.

We move now to ways of obtaining realizations of these abstract descriptions of consequence relations, beginning with the semantical structures called *logical matrices*.

2.3.2. Logical matrices and entailment relations

A *non-deterministic Σ-matrix*, or simply Σ*-nd-matrix*, is a structure $\mathbb{M} := \langle \mathbf{A}, D \rangle$, where $\mathbf{A}$ is a Σ-nd-algebra, whose carrier is the set of *truth-values*, and $D \subseteq A$ is the set of *designated truth-values*. In general, whenever $X \subseteq A$, we denote $A \backslash X$ by $\overline{X}$. In case $\mathbf{A}$ is deterministic, we simply say that $\mathbb{M}$ is a Σ*-matrix*. Also, $\mathbb{M}$ is *finite* when $\mathbf{A}$ is finite. When talking about $\mathbb{M}$, we may write $\mathsf{Val}(\mathbb{M})$ to refer to $\mathsf{Val}(\mathbf{A})$, the set of $\mathbb{M}$*-valuations*. Every Σ-nd-matrix $\mathbb{M}$ induces a substitution-invariant Set-Set consequence relation over Σ, denoted by $\rhd^{\mathsf{t}}_{\mathbb{M}}$, such that

$$\Phi \rhd^{\mathsf{t}}_{\mathbb{M}} \Psi \text{ if, and only if, for all } v \in \mathsf{Val}(\mathbb{M}), v[\Phi] \cap \overline{D} \neq \varnothing \text{ or } v[\Psi] \cap D \neq \varnothing \qquad (2.3)$$

The SET-FMLA companion of $\rhd^{\mathsf{t}}_{\mathbb{M}}$ will be denoted by $\models^{\mathsf{t}}_{\mathbb{M}}$. It is known that, when $\mathbf{A}$ is finite, $\rhd^{\mathsf{t}}_{\mathbb{M}}$ is finitary [57]. The above definition, when read in implicative form, tells us that, if $\Phi \rhd^{\mathsf{t}}_{\mathbb{M}} \Psi$, then, for all $\mathbb{M}$-valuations, at least one formula in Ψ must be assigned to a designated value in case every formula in Φ is assigned to a designated value. It is often more useful to consider the compatibility relation $\blacktriangleright^{\mathsf{t}}_{\mathbb{M}}$, according to which

$$\Phi \blacktriangleright^{\mathsf{t}}_{\mathbb{M}} \Psi \text{ if, and only if, for some } v \in \mathsf{Val}(\mathbb{M}), v[\Phi] \subseteq D \text{ and } v[\Psi] \subseteq \overline{D} \tag{2.4}$$

A valuation satisfying the right-hand side condition above is usually called a *countermodel* for the statement $\Phi \succ \Psi$ in $\mathbb{M}$.

It is worth mentioning that, in this work, we call non-deterministic matrices what in the literature is commonly referred to as *partial non-deterministic matrices* (or "PNmatrices" [7], for short). That is, non-deterministic matrices (or "Nmatrices") are usually assumed to be total (meaning that their algebraic counterparts are total) [2]. We perform this change due to the fact that all the results we present in this work apply to this more general setting, in view of the extension of the property of effectiveness (recall Theorem 5) to the partial case.

Let $\mathcal{M} := \{\mathbb{M}_i\}_{i \in I}$ be a family of Σ-nd-matrices. The SET-SET and SET-FMLA consequence relations associated to $\mathcal{M}$ are given, respectively, by $\models^{\mathsf{t}}_{\mathcal{M}} := \bigcap_{i \in I} \models^{\mathsf{t}}_{\mathbb{M}_i}$ and $\rhd^{\mathsf{t}}_{\mathcal{M}} := \bigcap_{i \in I} \rhd^{\mathsf{t}}_{\mathbb{M}_i}$. Given a SET-SET consequence relation $\rhd^{\mathsf{t}}$ and a SET-FMLA consequence relation $\models^{\mathsf{t}}$, we say that $\mathcal{M}$ *characterizes* $\rhd^{\mathsf{t}}$ when $\rhd^{\mathsf{t}} = \rhd^{\mathsf{t}}_{\mathcal{M}}$ and that $\mathcal{M}$ *characterizes* $\models^{\mathsf{t}}$ when $\models^{\mathsf{t}} = \models^{\mathsf{t}}_{\mathcal{M}}$. In both frameworks, we have the fundamental result, proved first by Wójcicki in 1970 [63] for SET-FMLA and then for SET-SET [24], stating that every substitution-invariant consequence relation over Σ is characterized by a family of Σ-matrices:

Theorem 6. *Every substitution-invariant consequence relation, being it* SET-FMLA *or* SET-SET*, is characterized by a family of Σ-matrices.*

In other words, substitution-invariant consequence relations over Σ cannot escape from having a semantics based on Σ-matrices. Whether a single matrix is enough depends on the verification of other properties on the concerned consequence relations, like *cancellation*, necessary and sufficient for the SET-FMLA case, plus *stability*, for the SET-SET case. The reader is referred to [24] for the definitions of these properties and the proofs of these results. In the next subsection, we will see that Theorem 6 is one of the ingredients for proving that consequence relations cannot avoid having another kind of semantics, this time causing much more surprise and being responsible for raising important questions about the project of many-valued logics.

2.3.3. Suszko's thesis

We begin by introducing a general kind of semantics, which is more liberal than Σ-nd-matrices in not demanding any kind of algebraic character or imposing any condition on the underlying valuations. A *valuation semantics over* Σ is a structure $\mathsf{S} := \langle \{v_i : L_\Sigma(P) \to \mathcal{V}_i\}_{i\in I}, \{\mathcal{D}_i\}_{i\in I}\rangle$, where $\mathcal{D}_i \subseteq \mathcal{V}_i$, for each $i \in I$, is the set of *designated* values associated to v_i. Both components of S are families whose set I of indices is assumed to be arbitrary. Notice that every Σ-nd-matrix $\mathbb{M}$ induces a valuation semantics $\mathsf{S}_\mathbb{M}$ where the family of valuations have as members the elements of $\mathsf{Val}(\mathbb{M})$ and the sets of designated elements are the set of designated values of $\mathbb{M}$.

A valuation semantics S induces a SET-SET consequence relation $\rhd^{\mathsf{t}}_{\mathsf{S}}$ in a similar way as for the case of Σ-nd-matrices presented in the previous section:

$$\Phi \rhd^{\mathsf{t}}_{\mathsf{S}} \Psi \text{ if, and only if, for all } i \in I, v_i[\Phi] \cap \overline{\mathcal{D}_i} \neq \varnothing \text{ or } v_i[\Psi] \cap \mathcal{D}_i \neq \varnothing \tag{2.5}$$

In case $\mathcal{V}_i$ has only and the same two elements for all $i \in I$, we say that S is a *bivaluation semantics*. A surprising result is that consequence relations, besides not escaping from a matrix-based semantics (Theorem 6), cannot avoid having a bivaluation semantics. This fact was used by Roman Suszko [61] to justify the thesis according to which there are only two logical values: the True ($\mathbf{t}$) and the False ($\mathbf{f}$). The proof is straightforward, relying on the so-called *Suszko's theorem*, given in Theorem 7 below, which produces a bivaluation semantics for a given family of matrices using the valuations and designated sets of those very matrices (a procedure usually referred to as *Suszko's reduction*).

Theorem 7 (Suszko's theorem [61])**.** *Every family $\mathcal{M}$ of Σ-matrices has a characteristic bivaluation semantics.*

Corollary 8. *Every substitution-invariant consequence relation, being it* SET-FMLA *or* SET-SET*, has a characteristic bivaluation semantics.*

In view of the above result, it seems hard to oppose to Suszko's thesis: there is always a shadow of bivalence behind many-valuedness, at least when we talk about consequence relations. We say, for this reason, that consequence relations are *inferentially two-valued*. We will see, however, that *inferentially many-valuedness* is possible: we only have to work with other notions of (one-dimensional or two-dimensional) logics.

2.3.4. q-consequences and p-consequences

The characteristics of Σ-matrices (and their induced consequence relations) that makes the proof of Theorem 7 work so smoothly can be easily spotted. First, the collection of truth-values is bipartitioned into a set of designated truth-values and a set of non-designated truth-values. Second, the associated consequence relation is defined in terms of preservation of designatedness. A natural way to begin a quest for inferential many-valuedness would be to modify some of those characteristics.

One could first try to change the notion of logic induced by a Σ-matrix $\mathbb{M}$ by considering the preservation of non-designated values:

$$\Phi \rhd^{\mathsf{f}}_{\mathbb{M}} \Psi \text{ if, and only if, for all } v \in \mathsf{Val}(\mathbb{M}),\, v[\Phi] \cap D \neq \varnothing \text{ or } v[\Psi] \cap \overline{D} \neq \varnothing \tag{2.6}$$

It is easy to check, however, that this logic is also a consequence relation and, thus, still inferentially two-valued. Notably, there is not much space left for modifications on the notion of logic in a Σ-nd-matrix, due to the bipartition of the set of truth-values. We have to follow another path if we want to be in the presence of (at least) a third logical value.

Following the work by G. Malinowski [42] on q-consequence operators ('q' is for *quasi*), we may achieve this goal by generalizing the notion of Σ-nd-matrices by getting rid of the bipartition of the set of truth-values, thus modifying the geometry of the matrix structure. Define a *Σ-nd-q-matrix* to be a structure $\mathfrak{Q} := \langle \mathbf{A}, \mathsf{Y}, \mathsf{N} \rangle$, where $\mathbf{A}$ is a Σ-nd-algebra, and $\mathsf{Y}, \mathsf{N} \subseteq A$ are disjoint sets of *designated* and *antidesignated* truth-values, intuitively read as values representing acceptance and rejection, respectively. For convenience, we let $\curlywedge := A \backslash \mathsf{Y}$ and $\mathsf{И} := A \backslash \mathsf{N}$, the sets of *non-designated* and *non-antidesignated* truth-values respectively. As with Σ-matrices, we set $\mathsf{Val}(\mathfrak{Q}) := \mathsf{Val}(\mathbf{A})$.

Surely, if we remain with the designatedness-preserving or non-designatedness-preserving (or even antidesignatedness-preserving or non-antidesignatedness-preserving) notion of consequence, we will end up with a consequence relation as before. In a Σ-nd-q-matrix, however, we have other options, as rejecting does not coincide with non-accepting. Investing on the interaction between acceptance and rejection is the way to go. Consider the one-dimensional Set-Set logic $\rhd^{\mathsf{q}}_{\mathfrak{Q}}$, called *$q$-entailment*, defined in the following way [13] [1]:

$$\Phi \rhd^{\mathsf{q}}_{\mathfrak{Q}} \Psi \text{ if, and only if, for all } v \in \mathsf{Val}(\mathfrak{Q}),\, v[\Phi] \cap \mathsf{N} \neq \varnothing \text{ or } v[\Psi] \cap \rotatebox[origin=c]{180}{Y} \neq \varnothing \tag{2.7}$$

The corresponding compatibility relation $\blacktriangleright^{\mathsf{q}}_{\mathfrak{Q}}$ is then given by:

$$\Phi \blacktriangleright^{\mathsf{q}}_{\mathfrak{Q}} \Psi \text{ if, and only if, for some } v \in \mathsf{Val}(\mathfrak{Q}),\, v[\Phi] \subseteq \reflectbox{N} \text{ and } v[\Psi] \subseteq \rotatebox[origin=c]{180}{Y} \tag{2.8}$$

From the latter, it is clear that **(O)** is not satisfied in general, as there might be values in $\rotatebox[origin=c]{180}{Y} \cap \reflectbox{N}$. We may also read this notion in implicative form: Φ follows from Ψ in case at least one of the formulas in Ψ is accepted whenever all formulas in Φ are non-rejected. The Set-Fmla companion of $\rhd^{\mathsf{q}}_{\mathfrak{Q}}$ is denoted by $\vdash_{\mathfrak{Q}}$.

From an abstract point of view, the notion of consequence just defined is a concretization of what are called q-consequence relations. As originally presented in [42], a Set-Fmla *q-consequence relation* is a relation $\vdash^{\mathsf{q}} \subseteq \mathsf{Pow}(L_\Sigma(P)) \times L_\Sigma(P)$ satisfying **(M)** and **(S)** plus a weakened version of transitivity, called *cumulative transitivity*:

(**Tq**) if $\Phi, \{\varphi : \Phi \vdash^{\mathsf{q}} \varphi\} \vdash^{\mathsf{q}} \psi$, then $\Phi \vdash^{\mathsf{q}} \psi$

In the presence of **(M)**, however, the above property turns to be equivalent to **(T$^+$)**. The Set-Set *q-consequence relations* were introduced in [12, 13] and consist in relations $\rhd^{\mathsf{q}} \subseteq \mathsf{Pow}(L_\Sigma(P)) \times \mathsf{Pow}(L_\Sigma(P))$ satisfying **(D)** and **(S$_\mathbf{S}$)**, plus

(**Cq**) if, for some $\Gamma \subseteq L_\Sigma(P)$, $\Pi, \Phi \rhd^{\mathsf{q}} \Psi, \Pi^{\mathsf{c}}$ for every $\Pi \subseteq \Gamma$, then $\Phi \rhd^{\mathsf{q}} \Psi$

Such property can be shown to be equivalent to **(C)**, whenever **(D)** is available.

A version of Wójcicki's result (Theorem 6) establishes the strict correspondence between Σ-q-matrices and q-consequence relations:

Theorem 9. *[42, Section 4] Every substitution-invariant q-consequence relation, being*

[1] We should point out that the notions related to q-consequence relations and p-consequence relations, to be presented in the sequel, were originally defined in [42, 27] for the Set-Fmla framework, and, in [13], they were extended to the Set-Set framework.

it SET-FMLA *or* SET-SET*, is characterized by a family of* Σ*-q-matrices.*

Similarly as we did for Σ-matrices, we may define a *valuation q-semantics over* Σ as a structure $\mathsf{S} := \langle \{v_i : L_\Sigma(P) \to \mathcal{V}_i\}_{i \in I}, \{\mathsf{Y}_i\}_{i \in I}, \{\mathsf{N}_i\}_{i \in I} \rangle$, where $\mathsf{Y}_i, \mathsf{N}_i \subseteq \mathcal{V}_i$ and $\mathsf{Y}_i \cap \mathsf{N}_i = \varnothing$, for each $i \in I$. We define then

$$\Phi \rhd^{\mathsf{q}}_{\mathsf{S}} \Psi \text{ if, and only if, for all } i \in I, v_i[\Phi] \cap \mathsf{N}_i \neq \varnothing \text{ or } v_i[\Psi] \cap \curlywedge_i \neq \varnothing \tag{2.9}$$

When $\mathcal{V}_i$ has only and the same three elements for all $i \in I$, we say that S is a *trivaluation q-semantics*. Two logical values are not enough to capture q-consequence relations in general. As proved in [42, 12], we have reached a scenario of inferential many-valuedness:

Theorem 10. *[42, Section 4] Every family* $\mathcal{M}$ *of* Σ*-q-matrices has a characteristic trivaluation semantics.*

Corollary 11. *Every substitution-invariant q-consequence relation, being it* SET-FMLA *or* SET-SET*, has a characteristic trivaluation semantics.*

One may argue that q-consequences do not represent genuine logics because they do not necessarily validate reflexivity (properties **(R)** and **(O)** for SET-FMLA and SET-SET, respectively). Indeed, looking through the lenses of preservation of validity from the premises to the conclusion, not concluding that p is valid when p is among the formulas presumed to be valid is unexpected, hard to justify. However, q-consequences can be seen as representing a form of hypothetical reasoning: when all formulas in the conclusion are not accepted, then at least one premise (here understood as a hypothesis) must be rejected.

Another inferentially many-valued one-dimensional notion of logic arises when we abandon transitivity instead of reflexivity. Introduced by S. Frankowski [27], the *p-consequence relations* ('*p*' for *plausible*) are, in SET-FMLA, those relations $\vdash^{\mathsf{p}} \subseteq \mathsf{Pow}(L_\Sigma(P)) \times L_\Sigma(P)$ satisfying **(R)** and **(M)**, or, in SET-SET, those relations $\rhd^{\mathsf{p}} \subseteq$

$\mathsf{Pow}(L_\Sigma(P)) \times \mathsf{Pow}(L_\Sigma(P))$ satisfying **(O)** and **(D)**. As with q-consequences, one way of obtaining realizations of p-consequences is by means of a matrix structure with a different geometry.

Define a Σ-*nd-p-matrix* to be a structure $\mathfrak{P} := \langle \mathbf{A}, \mathsf{Y}, \text{И} \rangle$, where $\mathbf{A}$ is a Σ-nd-algebra, and $\mathsf{Y} \subseteq \text{И} \subseteq A$ are the sets of *designated* and *non-antidesignated* or *plausible* truth-values. We let $\mathsf{Val}(\mathfrak{P}) := \mathsf{Val}(\mathbf{A})$. In this case, the associated one-dimensional Set-Set logic $\rhd^{\mathsf{p}}_{\mathfrak{P}}$, called *p-entailment*, is defined in the following way [13]:

$$\Phi \rhd^{\mathsf{p}}_{\mathfrak{P}} \Psi \text{ if, and only if, for all } v \in \mathsf{Val}(\mathfrak{P}), v[\Phi] \cap \text{⅄} \neq \varnothing \text{ or } v[\Psi] \cap \text{И} \neq \varnothing \qquad (2.10)$$

The corresponding compatibility relation $\blacktriangleright^{\mathsf{p}}_{\mathfrak{P}}$ is then given by:

$$\Phi \blacktriangleright^{\mathsf{p}}_{\mathfrak{P}} \Psi \text{ if, and only if, for some } v \in \mathsf{Val}(\mathfrak{P}), v[\Phi] \subseteq \mathsf{Y} \text{ and } v[\Psi] \subseteq \mathsf{N} \qquad (2.11)$$

The motivation for p-consequence relations is that of allowing for the loss of some degree of certainty (of being "certainly true") when passing from the premises to the conclusion. The definition of p-entailment just given makes this feature clear: valid inferences are those that permit concluding non-false from true propositions. We may formulate and prove (cf. [13]) versions of Theorem 9 and Corollary 10, which, when combined, give us that p-consequence relations are, as q-consequence relations, also (at most) inferentially three-valued. One can actually prove that the class of Σ-q-nd-matrices is isomorphic to the class of Σ-p-nd-matrices [13], so we may define q-entailment and p-entailment over both kinds of matrices, as done in [28, 58].

In this section, inferentially three-valued notions of one-dimensional logics were presented, showing that the multiplication of logical values is possible and may be used in contexts involving hypothetical reasoning or plausibility. We will see in a moment another notion of inferentially many-valued logic, this time over a two-dimensional framework, which constitutes an inferentially four-valued logic by itself and is also able to encompass, among others, the inferentially two-valued and three-valued notions we have seen.

2.4. Two-dimensional logics

Recall that logics in one dimension were defined as being SET-FMLA or SET-SET relations, with no *a priori* commitments to specific properties, like reflexivity (cf. **(R)** and **(O)**) or transitivity (cf. **(T)** or **(C)**). With a single dimension, we are not only able to express the preservation of truth (designatedness) or falsity (non-designatedness), which lead to inferentially two-valued logics, but also to represent other kinds of reasoning, for example, via q- and p-consequences, which are both inferentially (at most) three-valued notions of logic. Abstractly, this is achieved by abandoning some properties of consequence relations, and, semantically, by changing the geometry of logical matrices, with the notions of Σ-nd-q-matrices and Σ-nd-p-matrices.

Recent studies by C. Blasio [11, 12], J. Marcos and H. Wansing [13], with roots in the works of [41, 14], introduced a two-dimensional, inferentially four-valued notion of logical consequence worth studying from all typical lenses (abstractly, semantically and proof-theoretically), and also capable of capturing, in a unified environment, all the one-dimensional notions we have seen so far, in addition to many others. We will present in this section the main aspects related to the theory of such two-dimensional logics, focusing on the abstract characterizations and on the semantics of the two-dimensional semantical structures called B-*matrices*, here considered over non-deterministic algebras.

In order to provide an intuitive reading of SET^2-SET^2 statements, we shall adopt the notation $\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}}, \Phi_{\mathsf{И}}$, $\Phi_{\mathsf{⅄}}$ for arbitrary sets of formulas, instead of, respectively, $\Phi_{11}, \Phi_{21}, \Phi_{22}, \Phi_{12}$. We read the formulas in Φ_{Y} as being *accepted*; those in Φ_{N} as being *rejected*; those in $\Phi_{\mathsf{⅄}}$ as being *non-accepted*; and those in $\Phi_{\mathsf{И}}$ as being *non-rejected*. In this direction, SET^2-SET^2 statements may be read as asserting that, when all formulas in Φ_{Y} are accepted and all formulas in Φ_{N} are rejected, then either at least one formula in $\Phi_{\text{⅄}}$ is accepted or at least one formula in $\Phi_{\text{И}}$ is rejected.

We will use $\alpha, \beta, \gamma, \ldots$ to refer to elements in the set $\{\mathsf{Y}, \text{⅄}, \mathsf{N}, \text{И}\}$. Also, we will use $\tilde{\alpha}$ to refer to the flipped counterpart of α; that is, $\tilde{\mathsf{Y}} = \text{⅄}$, $\tilde{\text{⅄}} = \mathsf{Y}$, $\tilde{\mathsf{N}} = \text{И}$ and $\tilde{\text{И}} = \mathsf{N}$. Sometimes these symbols are used to denote sets of distinguished truth-values, as the reader might check in previous and subsequent sections of this document. The context will free these different usages from any ambiguity. It is worth pointing out that these symbols have been used in the literature to denote *cognitive attitudes* [11], which may be defined as positions that an agent is allowed to take with respect to a certain informational content. We consider that such an agent is allowed to take the positions of *acceptance* (Y), *non-acceptance* (⅄), *rejection* (N) or *non-rejection* (И) with respect to a given informational content. This agent may also mix positions of acceptance and rejection, for example, by both accepting *and* rejecting, both non-accepting *and* non-rejecting, both accepting *and* non-rejecting or both non-accepting *and* rejecting an informational content. These four possibilities are depicted in the bilattice in Figure 2.1 (cf. [13]), where the vertices represent the possible combinations of the cognitive attitudes, which, in turn, are placed on the edges. The values residing in the vertices are *logical values* determined by the cognitive attitudes [11]. This bilattice was used in [13] to prove that the two-dimensional notion of consequence we are about to meet is inferentially four-valued.

billatice-pic

Figure 2.1.: A bilattice representing the cognitive attitudes on the edges and logical values in the vertices, the latter emerging from combinations of the respective adjacent cognitive attitudes.

Cognitive attitudes, as explained above, can be seen as a bilateralist foundation to reason over propositions, where acceptance and rejection constitute two independent, although interacting, dimensions. In the next subsection, we will present the family of two-dimensional logics we will be interested in, which provide us with an appropriate logical environment to work with these bilateralist judgments.

2.4.1. B-consequence relations

A B-*consequence relation*[2] is a collection of SET^2-SET^2 statements, hereby called B-*statements*, according to which any of the following conditions constitute sufficient guarantee for the SET^2-SET^2 consequence judgment $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}}|\frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}$ to be established (recall that, given $\Phi \subseteq L_\Sigma(P)$, we denote by Φ^{c} the set $L_\Sigma(P)\backslash\Phi$):

(O2) $\Phi_{\mathsf{Y}} \cap \Phi_{\text{⅄}} \neq \varnothing$ or $\Phi_{\mathsf{N}} \cap \Phi_{\text{И}} \neq \varnothing$

(D2) $\dfrac{\Psi_{\text{И}}}{\Psi_{\mathsf{Y}}}\Big|\dfrac{\Psi_{\text{⅄}}}{\Psi_{\mathsf{N}}}$ and $\Psi_\alpha \subseteq \Phi_\alpha$ for every $\alpha \in \{\mathsf{Y}, \text{⅄}, \mathsf{N}, \text{И}\}$

(C2) $\dfrac{\Omega_{\text{Ƨ}}^{\mathsf{c}}}{\Omega_{\mathsf{S}}}\Big|\dfrac{\Omega_{\mathsf{S}}^{\mathsf{c}}}{\Omega_{\text{Ƨ}}}$ for all $\Phi_{\mathsf{Y}} \subseteq \Omega_{\mathsf{S}} \subseteq \Phi_{\text{⅄}}^{\mathsf{c}}$ and $\Phi_{\mathsf{N}} \subseteq \Omega_{\text{Ƨ}} \subseteq \Phi_{\text{И}}^{\mathsf{c}}$

These properties are easily perceived as generalizations to two-dimensional logics of the characteristic conditions of SET-SET consequence relations, namely **(O)**, **(D)** and **(C)**. Intuitively, **(O2)** establishes that a formula is accepted (resp. rejected) whenever it is present in the accepted (resp. rejected) formulas in the antecedent of the consecution (encoding, thus, a form of identity). In turn, **(D2)** states that a established consecution is not lost when new formulas are included in the antecedent and in the succedent of that very consecution (that is, B-consequence relations encode a form of monotonic reasoning). Finally, one can look at **(C2)** contrapositively to see one of its main effects: in case we are able to accept all formulas in Φ_{Y}, reject all formulas in Φ_{N}, non-accept all formulas in $\Phi_{\text{⅄}}$ and non-reject all formulas in $\Phi_{\text{И}}$, then we will be able to assign acceptance and rejection status to any other formula, covering the whole language in each dimension. This will, in particular, be useful in building countermodels for consecutions of interest in Chapter 5.

[2] We usually take "B" standing for Bilateralism, Bidimensional or Blasio [62].

The next result shows that any pair of unary predicates on Σ-formulas, each one seen as being related to a distinct dimension, induces a B-consequence relation defined in terms of preservation of at least one of the predicates along its corresponding dimension.

Lemma 12. *Let* $\mathsf{P}_{\mathsf{S}}, \mathsf{P}_{\mathsf{Ƨ}} : L_\Sigma(P) \to \{\mathbf{F}, \mathbf{T}\}$ *be predicates on* Σ*-formulas. Define the* 2×2*-place relation* $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}\mathsf{P}_{\mathsf{SƧ}}$ *by setting*

$$(\mathsf{P}\text{-}\mathbf{ent}) \quad \frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}\mathsf{P}_{\mathsf{SƧ}} \quad \textit{iff} \quad \begin{array}{l} \mathsf{P}_{\mathsf{S}}[\Phi_{\mathsf{Y}}] \subseteq \{\mathbf{T}\} \textit{ and } \mathsf{P}_{\mathsf{Ƨ}}[\Phi_{\mathsf{N}}] \subseteq \{\mathbf{T}\} \textit{ imply} \\ \mathsf{P}_{\mathsf{S}}[\Phi_{\mathsf{⅄}}] \not\subseteq \{\mathbf{F}\} \textit{ or } \mathsf{P}_{\mathsf{Ƨ}}[\Phi_{\mathsf{И}}] \not\subseteq \{\mathbf{F}\} \end{array}$$

or, in other words,

$$(\mathsf{P}\text{-}\mathbf{ent}) \quad \frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}\mathsf{P}_{\mathsf{SƧ}} \quad \textit{iff} \quad \begin{array}{l} \textit{it is not the case that} \\ \mathsf{P}_{\mathsf{S}}[\Phi_{\mathsf{Y}}] \subseteq \{\mathbf{T}\},\ \mathsf{P}_{\mathsf{Ƨ}}[\Phi_{\mathsf{N}}] \subseteq \{\mathbf{T}\}, \\ \mathsf{P}_{\mathsf{S}}[\Phi_{\mathsf{⅄}}] \subseteq \{\mathbf{F}\} \textit{ and } \mathsf{P}_{\mathsf{Ƨ}}[\Phi_{\mathsf{И}}] \subseteq \{\mathbf{F}\} \end{array}$$

Then the above relation is a B*-consequence relation.*

Proof. Suppose that $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\nmid\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}\mathsf{P}_{\mathsf{SƧ}}$, that is, (a) $\mathsf{P}_{\mathsf{S}}[\Phi_{\mathsf{Y}}] \subseteq \{\mathbf{T}\}$, $\mathsf{P}_{\mathsf{Ƨ}}[\Phi_{\mathsf{N}}] \subseteq \{\mathbf{T}\}$, $\mathsf{P}_{\mathsf{S}}[\Phi_{\mathsf{⅄}}] \subseteq \{\mathbf{F}\}$ and $\mathsf{P}_{\mathsf{Ƨ}}[\Phi_{\mathsf{И}}] \subseteq \{\mathbf{F}\}$. For **(O2)**, assume that $\varphi \in \Phi_{\mathsf{Y}} \cap \Phi_{\mathsf{⅄}}$. Then, by (a), we have $\mathsf{P}_{\mathsf{S}}(\varphi) = \mathbf{T}$ and $\mathsf{P}_{\mathsf{S}}(\varphi) = \mathbf{F}$, a contradiction. Similarly for the case $\varphi \in \Phi_{\mathsf{N}} \cap \Phi_{\mathsf{И}}$. For **(D2)**, assume that $\Psi_\alpha \subseteq \Phi_\alpha$, for each $\alpha \in \{\mathsf{Y}, \mathsf{⅄}, \mathsf{N}, \mathsf{И}\}$. Hence, (b): $\mathsf{P}_{\mathsf{S}}[\Psi_{\mathsf{Y}}] \subseteq \mathsf{P}_{\mathsf{S}}[\Phi_{\mathsf{Y}}]$, $\mathsf{P}_{\mathsf{S}}[\Psi_{\mathsf{⅄}}] \subseteq \mathsf{P}_{\mathsf{S}}[\Phi_{\mathsf{⅄}}]$, $\mathsf{P}_{\mathsf{Ƨ}}[\Psi_{\mathsf{N}}] \subseteq \mathsf{P}_{\mathsf{Ƨ}}[\Phi_{\mathsf{N}}]$ and $\mathsf{P}_{\mathsf{Ƨ}}[\Psi_{\mathsf{И}}] \subseteq \mathsf{P}_{\mathsf{Ƨ}}[\Phi_{\mathsf{И}}]$. By (a), (b) and transitivity of $\subseteq$, then, we obtain $\mathsf{P}_{\mathsf{S}}[\Psi_{\mathsf{Y}}] \subseteq \{\mathbf{T}\}$, $\mathsf{P}_{\mathsf{Ƨ}}[\Psi_{\mathsf{N}}] \subseteq \{\mathbf{T}\}$, $\mathsf{P}_{\mathsf{S}}[\Psi_{\mathsf{⅄}}] \subseteq \{\mathbf{F}\}$ and $\mathsf{P}_{\mathsf{Ƨ}}[\Psi_{\mathsf{И}}] \subseteq \{\mathbf{F}\}$, as desired. Finally, for **(C2)**, let $\Omega_{\mathsf{S}} := \{\varphi \in L_\Sigma(P) \mid \mathsf{P}_{\mathsf{S}}(\varphi) = \mathbf{T}\}$ and $\Omega_{\mathsf{Ƨ}} := \{\varphi \in L_\Sigma(P) \mid \mathsf{P}_{\mathsf{Ƨ}}(\varphi) = \mathbf{T}\}$. By (a), we have that $\Phi_{\mathsf{Y}} \subseteq \Omega_{\mathsf{S}} \subseteq \Phi_{\mathsf{⅄}}^{\mathsf{c}}$ and $\Phi_{\mathsf{N}} \subseteq \Omega_{\mathsf{Ƨ}} \subseteq \Phi_{\mathsf{И}}^{\mathsf{c}}$, and clearly $\frac{\Omega_{\mathsf{Ƨ}}^{\mathsf{c}}}{\Omega_{\mathsf{S}}}\nmid\frac{\Omega_{\mathsf{S}}^{\mathsf{c}}}{\Omega_{\mathsf{Ƨ}}}\mathsf{P}_{\mathsf{SƧ}}$, finishing the proof. □

A B-consequence relation is *substitution-invariant* if, in addition, $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}$ holds whenever:

(S2) $\frac{\Psi_{\mathsf{И}}}{\Psi_{\mathsf{Y}}}\Big|\frac{\Psi_{\mathsf{⅄}}}{\Psi_{\mathsf{N}}}$ and $\Phi_\alpha = \sigma(\Psi_\alpha)$ for every $\alpha \in \{\mathsf{Y}, \mathsf{⅄}, \mathsf{N}, \mathsf{И}\}$, for a substitution σ

Notice that we have chosen to present **(C2)** in a more compact formulation instead of the standard axioms introduced in [13] and written below:

(C$^{\mathbf{y}}$) given $\Psi \subseteq L_\Sigma(P)$, if $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}, \Omega_{\mathsf{S}}}\Big|\frac{\Phi_{\mathsf{⅄}}, \Psi\backslash\Omega_{\mathsf{S}}}{\Phi_{\mathsf{N}}}$ for all $\Omega_{\mathsf{S}} \subseteq \Psi$, then $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}$

$(\mathbf{C^n})$ given $\Psi \subseteq L_\Sigma(P)$, if $\dfrac{\Phi_{\text{И}}, \Psi\backslash\Omega_{\text{Ƨ}}}{\Phi_{\text{Y}}}\Big|\dfrac{\Phi_{\text{ʎ}}}{\Phi_{\text{И}}, \Omega_{\text{Ƨ}}}$ for all $\Omega_{\text{Ƨ}} \subseteq \Psi$, then $\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big|\dfrac{\Phi_{\text{ʎ}}}{\Phi_{\text{N}}}$

When we restrict the above properties to $L_\Sigma(P)$ (that is, taking the particular case in which Ψ above is $L_\Sigma(P)$), we have the following properties, called their *cut for* $L_\Sigma(P)$ versions:

$(\mathbf{C^y_L})$ if $\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}, \Omega_{\text{S}}}\Big|\dfrac{\Phi_{\text{ʎ}}, \Omega^c_{\text{S}}}{\Phi_{\text{N}}}$ for all $\Omega_{\text{S}} \subseteq L_\Sigma(P)$, then $\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big|\dfrac{\Phi_{\text{ʎ}}}{\Phi_{\text{N}}}$

$(\mathbf{C^n_L})$ if $\dfrac{\Phi_{\text{И}}, \Omega^c_{\text{Ƨ}}}{\Phi_{\text{Y}}}\Big|\dfrac{\Phi_{\text{ʎ}}}{\Phi_{\text{N}}, \Omega_{\text{Ƨ}}}$ for all $\Omega_{\text{Ƨ}} \subseteq L_\Sigma(P)$, then $\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big|\dfrac{\Phi_{\text{ʎ}}}{\Phi_{\text{N}}}$

It follows, then, that:

Proposition 13. *Properties* $(\mathbf{C^y})$ *and* $(\mathbf{C^n})$ *are equivalent, respectively, to* $(\mathbf{C^y_L})$ *and* $(\mathbf{C^n_L})$. *The latter pair of properties, in turn, is equivalent to* **(C2)**.

Proof. Notice that $(\mathbf{C^y})$ directly implies $(\mathbf{C^y_L})$, by taking Ψ as $L_\Sigma(P)$. Conversely, let $\Psi \subseteq L_\Sigma(P)$ and suppose that $\frac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\nmid\frac{\Phi_{\text{ʎ}}}{\Phi_{\text{N}}}$. Then, by $(\mathbf{C^y_L})$, there is a $\Omega_{\text{S}} \subseteq L_\Sigma(P)$ such that (a) $\frac{\Phi_{\text{И}}}{\Phi_{\text{Y}},\Omega_{\text{S}}}\nmid\frac{\Phi_{\text{ʎ}},\Omega^c_{\text{S}}}{\Phi_{\text{N}}}$. By taking $\Omega_{\text{S}} := \Psi \cap \Omega_{\text{S}}$, we have that $\Omega_{\text{S}} \subseteq \Omega_{\text{S}}$ and $\Psi\backslash\Omega_{\text{S}} \subseteq \Omega^c_{\text{S}}$. Hence, from (a) and **(D2)**, we have $\frac{\Phi_{\text{И}}}{\Phi_{\text{Y}},\Omega_{\text{S}}}\nmid\frac{\Phi_{\text{ʎ}},\Psi\backslash\Omega_{\text{S}}}{\Phi_{\text{N}}}$, as desired. The equivalence between $(\mathbf{C^n})$ and $(\mathbf{C^n_L})$ can be proved similarly.

We proceed to prove the equivalence between **(C2)** and the standard axioms, in their cut for $L_\Sigma(P)$ versions. In order to see that **(C2)** implies $(\mathbf{C^y_L})$ and $(\mathbf{C^n_L})$, suppose that $\frac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\nmid\frac{\Phi_{\text{ʎ}}}{\Phi_{\text{N}}}$. Then, by **(C2)**, there are $\Phi_{\text{Y}} \subseteq \Omega_{\text{S}} \subseteq \Phi^c_{\text{ʎ}}$ and $\Phi_{\text{N}} \subseteq \Omega_{\text{Ƨ}} \subseteq \Phi^c_{\text{И}}$ such that $\frac{\Omega^c_{\text{Ƨ}}}{\Omega_{\text{S}}}\nmid\frac{\Omega^c_{\text{S}}}{\Omega_{\text{Ƨ}}}$. Since $\Phi_{\text{N}} \subseteq \Omega_{\text{Ƨ}}$ and $\Phi_{\text{И}} \subseteq \Omega^c_{\text{Ƨ}}$, by the contrapositive version of **(D2)** we have $\frac{\Phi_{\text{И}}}{\Omega_{\text{S}}}\nmid\frac{\Omega^c_{\text{S}}}{\Phi_{\text{N}}}$ and thus $\frac{\Phi_{\text{И}}}{\Phi_{\text{Y}},\Omega_{\text{S}}}\nmid\frac{\Phi_{\text{ʎ}},\Omega^c_{\text{S}}}{\Phi_{\text{N}}}$, because $\Phi_{\text{Y}} \subseteq \Omega_{\text{S}}$ and $\Phi_{\text{ʎ}} \subseteq \Omega^c_{\text{S}}$, proving $(\mathbf{C^y_L})$. We can prove $(\mathbf{C^n_L})$ analogously.

For the other direction, again contrapositively, suppose that $\frac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\nmid\frac{\Phi_{\text{ʎ}}}{\Phi_{\text{N}}}$. Then, by $(\mathbf{C^y_L})$, there is a $\Omega_{\text{S}} \subseteq L_\Sigma(P)$ such that $\frac{\Phi_{\text{И}}}{\Phi_{\text{Y}},\Omega_{\text{S}}}\nmid\frac{\Phi_{\text{ʎ}},\Omega^c_{\text{S}}}{\Phi_{\text{N}}}$. By **(O2)**, we have that $(\Phi_{\text{Y}} \cup \Omega_{\text{S}}) \cap (\Phi_{\text{ʎ}} \cup \Omega^c_{\text{S}}) = \varnothing$, which implies that $\Phi_{\text{Y}} \subseteq \Omega_{\text{S}}$ and $\Phi_{\text{ʎ}} \subseteq \Omega^c_{\text{S}}$, hence $\Phi_{\text{Y}} \subseteq \Omega_{\text{S}} \subseteq \Phi^c_{\text{ʎ}}$ and $\frac{\Phi_{\text{И}}}{\Omega_{\text{S}}}\nmid\frac{\Omega^c_{\text{S}}}{\Phi_{\text{N}}}$. Similarly, from this and $(\mathbf{C^n_L})$, we have that there exists a $\Phi_{\text{N}} \subseteq \Omega_{\text{Ƨ}} \subseteq \Phi^c_{\text{И}}$ such that $\frac{\Omega^c_{\text{Ƨ}}}{\Omega_{\text{S}}}\Big|\frac{\Omega^c_{\text{S}}}{\Omega_{\text{Ƨ}}}$, as desired. $\square$

The properties of (substitution-invariant) B-consequence relations discussed up to now can be easily seen to be preserved under arbitrary intersections, as the result below states.

Lemma 14. *Let* $\{\mathsf{C}_i\}_{i\in I}$ *be a family of* B*-consequence relations. Then* $\mathsf{C} := \bigcap_{i\in I} \mathsf{C}_i$ *is a* B*-consequence relation. If each* C_i *is substitution-invariant, then* C *is also substitution-invariant.*

A B-consequence relation is called *finitary* when it enjoys the property

(F2) if $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}$, then $\frac{\Phi^{\mathsf{f}}_{\mathsf{И}}}{\Phi^{\mathsf{f}}_{\mathsf{Y}}}\Big|\frac{\Phi^{\mathsf{f}}_{\mathsf{⅄}}}{\Phi^{\mathsf{f}}_{\mathsf{N}}}$, for some finite $\Phi^{\mathsf{f}}_{\alpha} \subseteq \Phi_{\alpha}$, for every $\alpha \in \{\mathsf{Y}, \mathsf{⅄}, \mathsf{N}, \mathsf{И}\}$

In finitary B-consequence relations, axiom **(C2)** is equivalent to the pair

($\mathbf{C}^{\mathbf{u}}_{\mathsf{S}}$) if, for some $\varphi \in L_{\Sigma}(P)$, $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}, \varphi}\Big|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}$ and $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{⅄}}, \varphi}{\Phi_{\mathsf{N}}}$, then $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}$

($\mathbf{C}^{\mathbf{u}}_{\mathsf{Ƨ}}$) if, for some $\varphi \in L_{\Sigma}(P)$, $\frac{\Phi_{\mathsf{И}}, \varphi}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}$ and $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}, \varphi}$, then $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}$

We will see now that the two-dimensional environment is rich enough to allow for many one-dimensional logics to coinhabit the same logical structure. In [13], the authors provide a detailed study of many different one-dimensional *aspects* associated to a B-consequence relation. Each aspect corresponds to a collection of B-statements representing one-dimensional consecutions of a specific format. Just to give a couple of examples, the so-called gt-aspect of $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}$ is the collection of all B-statements of the form $\left(\begin{array}{c:c} \varnothing & \Phi_{\mathsf{⅄}} \\ \hdashline \Phi_{\mathsf{Y}} & \varnothing \end{array}\right)$ such that $\frac{\varnothing}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{⅄}}}{\varnothing}$, and the gq-aspect of $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}$ is the collection of all B-statements of the form $\left(\begin{array}{c:c} \Phi_{\mathsf{И}} & \Phi_{\mathsf{⅄}} \\ \hdashline \varnothing & \varnothing \end{array}\right)$ such that $\frac{\Phi_{\mathsf{И}}}{\varnothing}\Big|\frac{\Phi_{\mathsf{⅄}}}{\varnothing}$. Here, we will consider directly SET-SET logics associated to a B-consequence relation, in the following way:

Definition 15. *Let* $\mathsf{C} := \frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}$ *be a* B*-consequence relation. Define, then, the following one-dimensional* SET-SET *logics:*

- $\Phi_{\mathsf{Y}} \rhd^{\mathsf{C}}_{\mathsf{t}} \Phi_{\mathsf{⅄}}$ *if, and only if,* $\frac{\varnothing}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{⅄}}}{\varnothing}$
- $\Phi_{\mathsf{N}} \rhd^{\mathsf{C}}_{\mathsf{f}} \Phi_{\mathsf{И}}$ *if, and only if,* $\frac{\Phi_{\mathsf{И}}}{\varnothing}\Big|\frac{\varnothing}{\Phi_{\mathsf{N}}}$
- $\Phi_{\mathsf{И}} \rhd^{\mathsf{C}}_{\mathsf{q}} \Phi_{\mathsf{⅄}}$ *if, and only if,* $\frac{\Phi_{\mathsf{И}}}{\varnothing}\Big|\frac{\Phi_{\mathsf{⅄}}}{\varnothing}$

- $\Phi_{\mathsf{Y}} \rhd^{\mathsf{C}}_{\mathsf{p}} \Phi_{\mathsf{N}}$ *if, and only if,* $\frac{\varnothing}{\Phi_{\mathsf{Y}}}|\frac{\varnothing}{\Phi_{\mathsf{N}}}$
- $\Phi_{\mathsf{Y}} \rhd^{\mathsf{C}}_{\mathsf{r}} \Phi_{\mathsf{И}}$ *if, and only if,* $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}|\frac{\varnothing}{\varnothing}$
- $\Phi_{\mathsf{N}} \rhd^{\mathsf{C}}_{\mathsf{a}} \Phi_{\mathsf{⅄}}$ *if, and only if,* $\frac{\varnothing}{\varnothing}|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}$

Clearly, $\rhd^{\mathsf{C}}_{\mathsf{t}}$ and $\rhd^{\mathsf{C}}_{\mathsf{f}}$ constitute Set-Set consequence relations. We say that $\rhd^{\mathsf{C}}_{\mathsf{t}}$ and $\rhd^{\mathsf{C}}_{\mathsf{f}}$ *inhabits*, respectively, *the* t*-aspect* and *the* f*-aspect of* C. Furthermore, as pointed out in [13], $\rhd^{\mathsf{C}}_{\mathsf{q}}$ and $\rhd^{\mathsf{C}}_{\mathsf{p}}$ constitute, respectively, a q-consequence and a p-consequence provided $\frac{\varnothing}{\varphi}|\frac{\varnothing}{\varphi}$ holds for all $\varphi \in L_{\Sigma}(P)$. In this case, we say that $\rhd^{\mathsf{C}}_{\mathsf{q}}$ and $\rhd^{\mathsf{C}}_{\mathsf{p}}$ *inhabits*, respectively, *the* q*-aspect* and *the* p*-aspect of* C. On the other hand, if $\frac{\varphi}{\varnothing}|\frac{\varphi}{\varnothing}$ hold instead for all formula φ, we obtain logics which are dual to the latter. Finally, we included $\rhd^{\mathsf{C}}_{\mathsf{r}}$ and $\rhd^{\mathsf{C}}_{\mathsf{a}}$ in the above definition for calling the attention to the fact that the aspects corresponding to the statements of the form $\left(\begin{smallmatrix}\Phi_{\mathsf{И}} & \varnothing \\ \Phi_{\mathsf{Y}} & \varnothing\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\varnothing & \Phi_{\mathsf{⅄}} \\ \varnothing & \Phi_{\mathsf{N}}\end{smallmatrix}\right)$ were never considered in the literature, even though it seems natural to do so and study them in separate (something we leave for future work).

2.4.2. B-matrices and B-entailment

A *non-deterministic* B*-matrix over* Σ, or simply Σ*-nd-*B*-matrix*, is a structure $\mathfrak{M} := \langle \mathbf{A}, \mathsf{Y}, \mathsf{N} \rangle$, where $\mathbf{A}$ is a Σ-nd-algebra, $\mathsf{Y} \subseteq A$ is the set of *designated values* and $\mathsf{N} \subseteq A$ is the set of *antidesignated values* of $\mathfrak{M}$. For convenience, we define $\mathsf{⅄} := A \backslash \mathsf{Y}$ to be the set of *non-designated values*, and $\mathsf{И} := A \backslash \mathsf{N}$ as the set of *non-antidesignated values* of $\mathfrak{M}$. In case $\mathbf{A}$ is finite, $\mathfrak{M}$ is said to be *finite*. Given $X \subseteq A$, the *sub-*Σ*-nd-*B*-matrix induced by* X is given by $\mathfrak{M}_X := \langle \mathbf{A}_X, \mathsf{Y} \cap X, \mathsf{N} \cap X \rangle$. The elements of $\mathsf{Val}(\mathbf{A})$ are dubbed $\mathfrak{M}$*-valuations.*

Let $\mathfrak{M} := \langle \mathbf{A}, \mathsf{Y}, \mathsf{N} \rangle$ be a Σ-nd-B-matrix. The B*-entailment relation induced by* $\mathfrak{M}$ is a 2×2-place relation $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}_{\mathfrak{M}}$ over $L_{\Sigma}(P)$ such that

$$(\mathsf{B}\text{-ent}) \quad \frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}\,\mathfrak{M} \quad \text{iff} \quad \begin{array}{l}\text{there is no } \mathfrak{M}\text{-valuation } v \text{ such that} \\ v(\Phi_{\alpha}) \subseteq \alpha \text{ for every } \alpha \in \{\mathsf{Y}, \mathsf{⅄}, \mathsf{N}, \mathsf{И}\},\end{array}$$

for every $\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}}, \Phi_{⅄}, \Phi_{И} \subseteq L_\Sigma(P)$. Whenever $\frac{\Phi_{И}}{\Phi_{\mathsf{Y}}}|\frac{\Phi_{⅄}}{\Phi_{\mathsf{N}}}{}_{\mathfrak{M}}$, we say that the B-statement $\left(\frac{\Phi_{И}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{⅄}}{\Phi_{\mathsf{N}}}\right)$ *holds* in $\mathfrak{M}$. An $\mathfrak{M}$-valuation that bears witness to $\frac{\Phi_{И}}{\Phi_{\mathsf{Y}}}\nmid\frac{\Phi_{⅄}}{\Phi_{\mathsf{N}}}{}_{\mathfrak{M}}$ is called a *counter-model* for $\left(\frac{\Phi_{И}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{⅄}}{\Phi_{\mathsf{N}}}\right)$ in $\mathfrak{M}$.

Proposition 16. *The* B*-entailment relation induced by a* Σ*-nd-*B*-matrix is a substitution-invariant* B*-consequence relation.*

Proof. Given an $\mathfrak{M}$-valuation v, consider the predicates $\mathsf{P}^v_{\mathsf{S}}(\cdot) := v(\cdot) \in \mathsf{Y}$ and $\mathsf{P}^v_{Ƨ}(\cdot) := v(\cdot) \in \mathsf{N}$. We can see that $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}{}_{\mathfrak{M}} = \bigcap\left\{\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}{}_{\mathsf{P}^v_{\mathsf{S}Ƨ}}\right\}_{v \in \mathsf{Val}(\mathfrak{M})}$, where each $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}{}_{\mathsf{P}^v_{\mathsf{S}Ƨ}}$ is defined as per Lemma 12 and, by this same result, is a B-consequence relation. By Lemma 14, then, $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}{}_{\mathfrak{M}}$ is a B-consequence relation, thus it remains to prove that it is substitution-invariant. For that, contrapositively, suppose that, for some $\sigma \in \mathsf{Subs}_\Sigma$, we have $\frac{\sigma[\Phi_{И}]}{\sigma[\Phi_{\mathsf{Y}}]}\nmid\frac{\sigma[\Phi_{⅄}]}{\sigma[\Phi_{\mathsf{N}}]}{}_{\mathfrak{M}}$. Then, for some $v \in \mathsf{Val}(\mathfrak{M})$, $\frac{v[\sigma[\Phi_{И}]]}{v[\sigma[\Phi_{\mathsf{Y}}]]}\nmid\frac{v[\sigma[\Phi_{⅄}]]}{v[\sigma[\Phi_{\mathsf{N}}]]}{}_{\mathfrak{M}}$. As $v \circ \sigma$ is itself an $\mathfrak{M}$-valuation, we are done. $\square$

In the next examples, let $\mathsf{Y}_4 := \{\top, \mathbf{t}\}$, $\mathsf{N}_4 := \{\top, \mathbf{f}\}$, and recall the Σ^{FDE}-nd-algebras described in Examples 1, 2 and 3.

Example 17. *Consider the* Σ^{FDE}*-nd-*B*-matrix* $\mathfrak{M}^I := \langle \mathbf{I}, \mathsf{Y}_4, \mathsf{N}_4 \rangle$ *(cf. Example 1). The* $\mathtt{t}$*-aspect of* $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}{}_{\mathfrak{M}^I}$ *is inhabited by the logic introduced in [4], which incorporates some principles on how a processor would be expected to deal with information about an arbitrary set of formulas.*

Example 18. *Consider the* Σ^{FDE}*-nd-*B*-matrix* $\mathfrak{M}^E := \langle \mathbf{E}, \mathsf{Y}_4, \mathsf{N}_4 \rangle$ *(cf. Example 2). The induced* B*-entailment corresponds to the logic* $\mathbf{E}^B$ *presented in [11], studied as a version of Dunn-Belnap's four-valued logic in which the informational content of sentences is also taken into account in inferences, instead of only the truth content.*

Example 19. *Consider the* Σ^{FDE}*-nd-*B*-matrix* $\mathfrak{M}^K := \langle \mathbf{K}, \mathsf{Y}_4, \mathsf{N}_4 \rangle$ *(cf. Example 3). As shown in [17], Kleene's strong three-valued logic inhabits the* $\mathtt{t}$*-aspect of* $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}{}_{\mathfrak{M}^K}$.

Example 20. *Let* $\mathcal{V}_5 := \{f, F, I, T, t\}$, $\mathsf{Y}_5 := \{T, I, t\}$, $\mathsf{N}_5 := \{T, I, f\}$*, and consider a signature* $\Sigma^{\mathbf{mCi}}$ *containing but three binary connectives,* $\wedge$*,* $\vee$ *and* $\supset$*, and two unary* *connectives, $\neg$ and $\circ$. Inspired by the 5-valued non-deterministic logical matrix presented in [1] for the logic of formal inconsistency called* **mCi** *[47] — to which the whole Chapter 6 is devoted —, we define the $\Sigma^{\mathbf{mCi}}$-nd-B-matrix $\mathfrak{M}_{\mathbf{mCi}} := \langle \mathbf{A}_5, \mathsf{Y}_5, \mathsf{N}_5 \rangle$ with the following interpretations:*

$$\wedge_{\mathbf{A}_5}(x_1, x_2) := \begin{cases} \{f\} & \textit{if either } x_1 \notin \mathsf{Y}_5 \textit{ or } x_2 \notin \mathsf{Y}_5 \\ \{t, I\} & \textit{otherwise} \end{cases}$$

$$\vee_{\mathbf{A}_5}(x_1, x_2) := \begin{cases} \{t, I\} & \textit{if either } x_1 \in \mathsf{Y}_5 \textit{ or } x_2 \in \mathsf{Y}_5 \\ \{f\} & \textit{if } x_1, x_2 \notin \mathsf{Y}_5 \end{cases}$$

$$\supset_{\mathbf{A}_5}(x_1, x_2) := \begin{cases} \{t, I\} & \textit{if either } x_1 \notin \mathsf{Y}_5 \textit{ or } x_2 \in \mathsf{Y}_5 \\ \{f\} & \textit{if } x_1 \in \mathsf{Y}_5 \textit{ and } x_2 \notin \mathsf{Y}_5 \end{cases}$$

	f	F	I	T	t
$\neg_{\mathbf{A}_5}$	t,I	T	t,I	F	f

	f	F	I	T	t
$\circ_{\mathbf{A}_5}$	T	T	F	T	T

We note that the logic **mCi** *inhabits the* t*-aspect of* $\cdot|\cdot\, \mathfrak{M}_{\mathbf{mCi}}$ *. It is worth pointing out that, up to now, no* finite *Hilbert-style calculus was known to axiomatize this logic; however, a finite two-dimensional symmetrical Hilbert-style calculus (see the related definitions in Section 2.5.2) for* **mCi** *results smoothly from the procedure described in Chapter 5.*

We have seen that one-dimensional consequence relations are inferentially two-valued, and the price to pay for inferentially many-valuedness is to abandon some axioms which one may regard as essential for a logic, like reflexivity and transitivity. In two dimensions, however, inferential many-valuedness is built in the very notion of B-consequence relation, whose axioms are generalizations of those of one-dimensional consequence relations. This result obtains from generalizations of Theorems 6 and 7, which we present in the next two results. Detailed proofs may be found in [13].

Theorem 21. *Every substitution-invariant* B*-consequence relation is characterized by a family of* Σ-B*-matrices.*

In a similar way as the definitions of valuation semantics, q-semantics and p-semantics, which generalize the corresponding semantics of matrices, we define a *valuation* B*-semantics* as a structure $\mathsf{S} := \langle \{v_i : L_\Sigma(P) \to \mathcal{V}_i\}_{i \in I}, \{\mathsf{Y}_i\}_{i \in I}, \{\mathsf{N}_i\}_{i \in I} \rangle$, where $\mathsf{Y}_i, \mathsf{N}_i \subseteq \mathcal{V}_i$, for each $i \in I$, are sets of *designated* and *antidesignated* values associated to the valuation v_i. We adopt also here the notation $\mathsf{\lambda}_i$ and $\mathsf{И}_i$ to refer to the sets $\mathcal{V}_i \backslash \mathsf{Y}_i$ and $\mathcal{V}_i \backslash \mathsf{N}_i$, respectively. We may easily check that S induces a B-consequence relation $\vdots|\vdots_{\mathsf{S}}$ such that

$$(\mathsf{B}\text{-sem}) \quad \frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}} \Big| \frac{\Phi_{\mathsf{\lambda}}}{\Phi_{\mathsf{N}}}\,\mathsf{S} \quad \text{iff} \quad \begin{array}{l} \text{there is no } i \in I \text{ such that} \\ v_i(\Phi_\alpha) \subseteq \alpha_i \text{ for every } \alpha \in \{\mathsf{Y}, \mathsf{\lambda}, \mathsf{N}, \mathsf{И}\}, \end{array}$$

for every $\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}}, \Phi_{\mathsf{\lambda}}, \Phi_{\mathsf{И}} \subseteq L_\Sigma(P)$. When $|\mathcal{V}_i| = 4$ for all $i \in I$, S is said to be a *tetra-valuation* B*-semantics*.

Then:

Theorem 22. *Every family of* Σ-B*-matrices has a tetra-valuation* B*-semantics.*

2.5. Deductive formalisms

Semantical structures confer meaning to the sentences of a given language by the way they assign truth-values to these very sentences. They provide means to obtain logics based on the compatibility of the qualities associated to those truth-values, like the qualities of being designated or antidesignated. An alternative to this semantical approach to logics is *Proof Theory*, which consists in the investigation of inferential mechanisms, the so-called deductive systems, that manipulate syntactical objects, like formulas and sequents, in order to explain in terms of step-by-step derivations why a certain statement holds according to a given logic. In many cases, this allows us to assign meaning to the connectives of a logic based on their inferential behaviour, something that occupies logicians working in the field of Philosophical Proof Theory (see [36, Chapter 4]). We formulate below general proof-theoretical notions that allows us to compare different classes of deductive systems.

Deductive formalisms are, in a broad sense, specifications of two kinds of objects: *rules of inference* and *derivations*. The former are usually collections of *rule instances*, which are specified in terms of the kind of syntactical objects they manipulate — formulas, sequents or B-sequents, for example — and the way we may identify their *antecedent* (those objects taken as inputs), and their *succedent* (those objects that are produced from the inputs). Derivations, in turn, have to be specified as structures that can be built somehow by applications of the rules of inference (or, as is usual, of their rule instances). We obtain *deductive systems* on top of a deductive formalism just by collecting specific rules of inference conforming to the specifications of that formalism. Deductive systems may be grouped into *deductive approaches*, according to general characteristics of their rules of inference. Moreover, deductive systems are expected to induce some notion of logic, specified as a collection of statements for which there are derivations witnessing the provability — defined according to some criterion — of these very statements in the system.

After this general account of deductive systems, we proceed to discuss some classes of deductive formalisms, allowing us to compare them later with the formalisms presented in future chapters.

2.5.1. G-formalisms

Popular deductive formalisms are those whose rules of inference are collections of $(n+1)$-ary tuples of *sequents*, which, for us, are objects of the form $\Phi \succ \psi$ (for Set-Fmla formalisms), $\Phi \succ \Psi$ (for Set-Set formalisms) or $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}} \| \frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}$ (for Set^2-Set^2 formalisms), where the involved sets of formulas are finite. More precisely, a rule of inference justifies a *(conclusion-)sequent*, the $(n+1)$-th element of the tuple, when taking a collection of *(premise-)sequents* (the elements occupying positions 1 to n in the tuple) as input. Such rules are usually specified schematically (that is, by a representative instance containing variables for formulas and sets of formulas, expected to be replaced by specific formulas when applied in a derivation) with a horizontal bar separating the premise-sequents (at the top) and the conclusion-sequent (at the bottom). We call such formalisms *G-formalisms*, where 'G' stands for 'Gentzen'. The associated deductive systems are called *G-systems* [3].

Example 23. *The G-formalism of* B*-sequents works as in the one-dimensional case, with the difference that rules manipulate* B*-sequents and derivations are built in order to show that a* B*-sequent of interest is provable. We limit ourselves here to present an example of a* B*-sequent calculus, which was proven in [11] to induce the same* B*-consequence relation as the one induced by the* Σ*-*B*-matrix* $\mathfrak{M}^E$*. Define* $\mathfrak{G}^E$ *to be the* B*-sequent calculus whose rules of inference are the following:*

Structural rules

$$\frac{}{\dfrac{}{\varphi}\Big\|\dfrac{\varphi}{}}\ in_{\mathsf{S}} \qquad \frac{}{\dfrac{\varphi}{}\Big\|\dfrac{}{\varphi}}\ in_{\text{Ƨ}}$$

$$\frac{\dfrac{\Phi_{\text{ᴎ}}}{\Phi_{\mathsf{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}}{\dfrac{\Phi_{\text{ᴎ}},\Psi_{\text{ᴎ}}}{\Phi_{\mathsf{Y}},\Psi_{\mathsf{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}},\Psi_{\text{⅄}}}{\Phi_{\mathsf{N}},\Psi_{\mathsf{N}}}}\ weak \qquad \frac{\dfrac{\Phi_{\text{ᴎ}}}{\Phi_{\mathsf{Y}},\varphi}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}} \qquad \dfrac{\Phi_{\text{ᴎ}}}{\Phi_{\mathsf{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}},\varphi}{\Phi_{\mathsf{N}}}}{\dfrac{\Phi_{\text{ᴎ}}}{\Phi_{\mathsf{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}}\ cut_{\mathsf{S}} \qquad \frac{\dfrac{\Phi_{\text{ᴎ}}}{\Phi_{\mathsf{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}},\varphi} \qquad \dfrac{\Phi_{\text{ᴎ}},\varphi}{\Phi_{\mathsf{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}}{\dfrac{\Phi_{\text{ᴎ}}}{\Phi_{\mathsf{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}}\ cut_{\text{Ƨ}}$$

[3] We understand that G-systems have, in general, more complex formulations than the one we provide informally here. The way we describe them, nevertheless, is enough to illustrate the differences with respect to Hilbert-style systems in the next subsection.

Logical rules

$$\dfrac{\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}},\varphi} \qquad \dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}},\psi}}{\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}},\varphi\wedge\psi}}\ L\wedge_{\text{Ƨ}} \qquad \dfrac{\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}},\varphi}{\Phi_{\text{N}}} \qquad \dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}},\psi}{\Phi_{\text{N}}}}{\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}},\varphi\wedge\psi}{\Phi_{\text{N}}}}\ R\wedge_{\text{S}}$$

$$\dfrac{\dfrac{\varphi,\psi,\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}{\dfrac{\Phi_{\text{И}},\varphi\wedge\psi}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}\ R\wedge_{\text{Ƨ}} \qquad \dfrac{\dfrac{\Phi_{\text{И}}}{\varphi,\psi,\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}{\dfrac{\Phi_{\text{И}}}{\varphi\wedge\psi,\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}\ L\wedge_{\text{S}}$$

$$\dfrac{\dfrac{\Phi_{\text{И}}}{\varphi,\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}} \qquad \dfrac{\Phi_{\text{И}}}{\psi,\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}{\dfrac{\Phi_{\text{И}}}{\varphi\vee\psi,\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}\ L\vee_{\text{S}} \qquad \dfrac{\dfrac{\varphi,\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}} \qquad \dfrac{\psi,\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}{\dfrac{\varphi\vee\psi,\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}\ R\vee_{\text{Ƨ}}$$

$$\dfrac{\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}},\varphi,\psi}{\Phi_{\text{N}}}}{\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}},\varphi\vee\psi}{\Phi_{\text{N}}}}\ R\vee_{\text{S}} \qquad \dfrac{\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}},\varphi,\psi}}{\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}},\varphi\vee\psi}}\ R\vee_{\text{Ƨ}}$$

$$\dfrac{\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}},\varphi}{\Phi_{\text{N}}}}{\dfrac{\neg\varphi,\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}\ R\neg_{\text{Ƨ}} \qquad \dfrac{\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}},\varphi}}{\dfrac{\Phi_{\text{И}}}{\neg\varphi,\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}\ L\neg_{\text{S}} \qquad \dfrac{\dfrac{\varphi,\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}{\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\neg\varphi,\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}\ R\neg_{\text{S}} \qquad \dfrac{\dfrac{\Phi_{\text{И}}}{\varphi,\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\Phi_{\text{N}}}}{\dfrac{\Phi_{\text{И}}}{\Phi_{\text{Y}}}\Big\|\dfrac{\Phi_{\text{⅄}}}{\neg\varphi,\Phi_{\text{N}}}}\ L\neg_{\text{Ƨ}}$$

Derivations in G-systems are sequences or trees of sequents, the presence of each sequent in the sequence being justified by an application of a rule of inference of the system taking as inputs previous sequents in the sequence. When a sequent happens to appear at the end of a derivation, we say that it is *provable*. We may also represent proofs as trees that grow upwards and have the provable sequent at the root. The one-dimensional Set-Fmla logic associated to a Set-Fmla G-system is the one according to which ψ follows from Φ whenever the sequent $\Phi \succ \psi$ is provable in that system, and it can be shown to be a substitution-invariant and finitary $\textsc{Set-Fmla}$ consequence relation. This extends to the $\textsc{Set-Set}$ and $\textsc{Set}^2\textsc{-Set}^2$ cases in the expected way. Two well-known deductive approaches when working with G-systems are the *Natural Deduction* and the *Gentzen Calculus* approaches. In the former, on the one hand, rules are focused on introducing and eliminating compound formulas on the right side of the sequents being manipulated. The latter, on the other hand, does not privilege a specific side, focusing on rules that introduce compound formulas either on the left or on the right.

Arguably the most distinguishing aspect of G-formalisms is that the rules manipulate meta-linguistic objects (the sequents) representing inferences by themselves, something that aggregates power to the formalism by allowing the manipulation of *contexts*, as well as the mechanism of *discharge of assumptions*. With these resources, it becomes easy to internalize in the calculus meta-properties of the corresponding logic. For example, the following rules internalize the *Deduction-Detachment Theorem*, read as $\Phi, \varphi \vdash^{\mathsf{t}} \psi$ iff $\Phi \vdash^{\mathsf{t}} \varphi \supset \psi$, a well-known meta-property of intuitionistic and classical logic:

$$\frac{\Phi, \varphi \succ \psi}{\Phi \succ \varphi \supset \psi} \qquad \frac{\Phi \succ \varphi \quad \Psi \succ \varphi \supset \psi}{\Phi, \Psi \succ \psi}$$

Such power, however, does not come for free: the price is a distancing of the form of the rules from the associated notions of logic, posing difficulties for some important investigations, like those concerning the correspondence between merging different deductive systems and combining their underlying logics. A way of realizing what we mean is by noticing that rules in G-systems do not correspond to statements of the logic they induce, that is, they do not consist simply of formulas or sets of formulas grouped in a tuple, they are more complex than that.

2.5.2. H-formalisms

While rules in G-formalisms manipulate meta-linguistic objects representing inferences by themselves, what we call *H-formalisms* ('H' is for 'Hilbert') provide deductive systems as *logical bases*, in the sense that the instances of the rules of inference are elements of the induced logics, and these very logics are the least logics closed under such rule instances. In other words, the type of the relation that represents the logics determines the shape of the rule instances.

For instance, since one-dimensional logics are relations $\vdash\subseteq \mathsf{Pow}(L_\Sigma(P)) \times L_\Sigma(P)$ (in Set-Fmla) or $\rhd\subseteq \mathsf{Pow}(L_\Sigma(P)) \times \mathsf{Pow}(L_\Sigma(P))$ (in Set-Set), rule instances are expected to be, respectively, pairs (Φ, ψ) or (Φ, Ψ), where the first component is the *antecedent* and the second, the *succedent* of the rule instance. Noticeably, these are simpler than the rule instances of G-formalisms, as they do not allow for the manipulation of contexts and the discharge of assumptions. Rules of inference are again usually presented schematically, and they are graphically represented with a horizontal bar, placing the antecedent at the top, and the succedent at the bottom:

$$\frac{\Phi}{\psi} \quad \frac{\Phi}{\Psi}$$

Traditionally, Hilbert deductive systems have been associated to a specific H-formalism, which we will call Set-Fmla *H-formalism.* The associated deductive systems will be called Set-Fmla *H-systems.* Rule instances, in this case, have the shape we have shown above (in Set-Fmla), and derivations (we shall call them Set-Fmla *derivations*) are finite sequences of formulas, where each formula in the sequence is either a premise or appears in the succedent of a rule instance whose antecedent formulas appear previously in the sequence. We say, in other words, that each (non-premise) formula derives or results from an application of a rule to previously derived formulas. Provided we have an account of the rules applied to produce each formula in a derivation, we may prefer to rewrite it as a tree instead of a sequence of formulas, where the conclusion appears in the root and premises appear in the leaves. Given a Set-Fmla H-system $\mathcal{H}$, we let $\Phi \vdash_{\mathcal{H}} \psi$ if, and only if, there is a Set-Fmla derivation ending in ψ, using only the rules of $\mathcal{H}$ and premises in Φ. The relation $\vdash_{\mathcal{H}}$ so defined can be easily proved to be a finitary and substitution-invariant[4] Set-Fmla consequence relation.

We should emphasize here that Set-Fmla H-systems (and H-formalisms in general) are committed neither to specific connectives (like implication) and rules of inference (like *Modus Ponens*), nor to having many axioms and few rules of inference, as some texts might make us believe [48]. If this were true, a quick inspection of the well-known Post's lattice [53, 33] would seem to show that there is an infinite number of 2-valued theoremless logics that would allegedly not be capturable by H-systems. The situation would of course be even worse if we considered many-valued logics such as Kleene's K3 or Dunn-Belnap's FDE, which also do not have theorems — and thus, no axioms — and no implication either.

Example 24. *An H-system for Dunn-Belnap's FDE in* Set-Fmla *is given by the following rule schemas [26]:*

$$\frac{p \wedge q}{p} \quad \frac{p \wedge q}{q} \quad \frac{p, q}{p \wedge q} \quad \frac{p}{p \vee q} \quad \frac{p \vee q}{q \vee p} \quad \frac{p \vee p}{p}$$

$$\frac{p \vee (q \vee r)}{(p \vee q) \vee r} \quad \frac{p \vee (q \wedge r)}{(p \vee q) \wedge (p \vee r)} \quad \frac{(p \vee q) \wedge (p \vee r)}{p \vee (q \wedge r)}$$

$$\frac{p \vee r}{\neg\neg p \vee r} \quad \frac{\neg\neg p \vee r}{p \vee r} \quad \frac{\neg(p \vee q) \vee r}{(\neg p \wedge \neg q) \vee r} \quad \frac{(\neg p \wedge \neg q) \vee r}{\neg(p \vee q) \vee r}$$

$$\frac{\neg(p \wedge q) \vee r}{(\neg p \vee \neg q) \vee r} \quad \frac{(\neg p \vee \neg q) \vee r}{\neg(p \wedge q) \vee r}$$

We may agree, however, with the general claim stating that Set-Fmla H-systems

[4]Assuming the system is *schematic*, namely, that each of its rule instances is a substitution instance of a representative statement, called *rule schema*.

are commonly impractical, difficult mechanisms, both during system design and during the production of derivations. The unavailability of modularity (a clear account of the effect of each rule over a given semantics) and analyticity results for these systems often demands non-intuitive or clever choices of rule instances to apply in derivations, and makes the automation of reasoning hard to achieve. As we will see in Chapter 3, in view of recent developments on Set-Set H-systems, this claim cannot be extended to H-formalisms in general.

2.5.3. Signed formalisms

As we know from Section 2.3.4, there are other notions of (one-dimensional) logics that do not satisfy the usual properties of reflexivity and transitivity, namely q-consequences and p-consequences. Deductive formalisms for them have scarcely been developed at all in the literature, a lack that we will attempt to fill with our very inclusive two-dimensional formalism to be presented in the next chapter.

We briefly comment now on a proof formalism associated to p-consequences, introduced by Frankowski in [29]. Rules of inference are called *p-rules of inference*, and their instances, called *p-inferences*, manipulate pairs from the product $L_\Sigma(P) \times \{+, *\}$, which we call *signed formulas*. The antecedent of a p-inference is a set of signed formulas, and the succedent is a single signed formula. We will denote a signed formula (φ, s) by φ^s, where $s \in \{+, *\}$, approximating thus this formalism to those used for signed consequences in the context of bilateralist (one-dimensional) logics [54, 23]. The intuition behind signed formulas is that the sign $+$ indicates that the formula is well justified, while $*$ indicates that it is plausible only. As expected, a *p-calculus* R is a collection of p-rules of inference. A *p-derivation* with set of premises Φ in R, then, is similar to those in the Set-Fmla H-formalism, but with signed formulas and a special way of incorporating premises: it is a sequence $\varphi_1^{s_1}, \ldots, \varphi_k^{s_k}$ of signed formulas where each $\varphi_i^{s_i}$

is such that either $\varphi_i \in \Phi$ and $s_i = +$ or there is an instance of a rule of inference of R whose antecedent is included in the set of previous formulas in the sequence and the succedent is $\varphi_i^{s_i}$. A *p-proof* of φ^s from Φ in R is a p-derivation whose last signed formula is φ^s. In order to connect this formalism with the p-consequence relation (whose statements involve formulas, not signed formulas), we say that φ is provable from Φ in R when there is a p-proof of φ^+ or φ^* from Φ. This formalism does not fit in the class of G-formalisms or H-formalisms with respect to p-consequences, as we have presented, since rules does not manipulate inferences and the calculus does not form a logical basis of the p-consequence relation. In the next chapter, however, we will see an H-formalism where signals may be translated to positions in a two-dimensional structure.

3. Symmetrical H-systems

As mentioned in Section 2.5.2, when we pass from Set-Fmla H-systems to Set-Set H-systems, the first notable difference is that succedents of rule instances become sets of formulas (including the empty set), instead of a single formula. This correspondence between the objects that constitute antecedents and succedents characterizes what we call *symmetrical H-systems*. The other difference one might expect in this passage is with respect to the derivations; in particular, how is a derivation affected by the multiple formulas in the succedent of an applied rule instance, and how do we know that we have obtained a proof of a Set-Set statement of interest? The answer to this, coming originally from [57], is that derivations become labelled rooted trees, where an application of a rule instance produces as many branches as formulas in the succedent.

The goal of this chapter is to present the Set-Set H-formalism, and, more importantly, to show that we can generalize its inner workings to produce symmetrical H-formalisms over other frameworks; in particular, over the Set^2-Set^2 framework, which will occupy us in the whole Chapter 4.

3.1. Derivations as rooted labelled trees

The structures called *rooted labelled trees* (or *trees*, for short) will be extensively used in the representation of derivations and proofs in the so-called Set-Set and Set^2-Set^2 H-formalisms, which are presented, respectively, in Section 3.2 and Chapter 4. Our purpose now is to give a general formal account of them in order to abbreviate some proofs in other parts of this work, as well as to show how derivations based on such trees might be adapted to manipulate more complex objects.

Before proceeding, let us introduce some notions from universal algebra and order theory. A partially ordered set (poset) is said to be *complete* when it has a least element and each of its chains has a supremum. Let $\mathbf{L}$ be a lattice with least element $\perp_{\mathbf{L}}$ and denote by $\leq_{\mathbf{L}}$ its underlying partial order. We say that $\mathbf{L}$ is *complete* whenever it is complete when seen as a poset. An element a of $\mathbf{L}$ is called an *atom* when there is no element in between $\perp_{\mathbf{L}}$ and a with respect to $\leq_{\mathbf{L}}$. When every element of $\mathbf{L}$ is the supremum of a collection of atoms, then $\mathbf{L}$ is said to be *atomistic*. For a complete introduction to lattice theory, see [8].

Throughout this section, let $\mathcal{L} := \left\langle L, \cdot^{c\mathcal{L}}, \sqcup_{\mathcal{L}}, \sqcap_{\mathcal{L}}, \top_{\mathcal{L}}, \perp_{\mathcal{L}} \right\rangle$ be a complete atomistic (we will use this property specially in the proof of Proposition 43) Boolean algebra, with underlying complete partial order denoted by $\leq^{\mathcal{L}}$. Note that we denote by $\cdot^{c\mathcal{L}}$ the complement operation in $\mathcal{L}$. Given $Y \subseteq L$ a chain with respect to $\leq^{\mathcal{L}}$, we denote the supremum of Y by $\bigsqcup Y$. This structure is said to be a *node labels algebra*, and is intended to provide the information carried out by the nodes of a tree. Also, let $*$ be an special symbol (not in L), called the *discontinuation symbol.* Moreover, call the elements in $L \times L$ *expansors over* $\mathcal{L}$, and fix a nonempty collection E of such elements. They are intended to represent ways of expanding trees during derivations, producing new branches with new pieces of information. We will usually write an expansor $e := (l, l') \in E$ as

$$\frac{l}{l'}.$$

Our intention is that the rule instances of a symmetrical H-system be particular cases of expansors. In view of that, as one would expect, we call l the *antecedent* and l' the *succedent* of e. The new information introduced by an application of an expansor will be determined by a function $\mathsf{exp} : L \to (\mathsf{Pow}(L) \cup \{*\})$, such that $\mathsf{exp}(\bot_{\mathcal{L}}) = *$ and, for all $l \neq \bot_{\mathcal{L}}$, $\mathsf{exp}(l)$ is a collection of atoms of $\mathcal{L}$ and $\bigsqcup \mathsf{exp}(l) = l$. Clearly, $\mathsf{exp}(l) \neq *$ for all $l \neq \bot_{\mathcal{L}}$. The output of this function when taking a label, thus, is either a discontinuation symbol or a set of labels. As we will detail in a moment, each of these labels will correspond to a new branch in the derivation where the expansor is applied, and each of these branches will contain all the information produced up to the application of the expansor, plus the information carried by the corresponding label.

Last but not least, we associate to $\mathcal{L}$ a binary predicate on L denoted by $\mathtt{match}$ such that $l \,\mathtt{match}\, l'$ if, and only if, $l \sqcap_{\mathcal{L}} l' \neq \bot_{\mathcal{L}}$. Notice that this predicate is monotonic on the second argument with respect to the order $\leq^{\mathcal{L}}$, and reflexive on labels different from $\bot_{\mathcal{L}}$. Furthermore, note that, whenever $l \,\mathtt{match}\, l'$, it follows that $\mathsf{exp}(l') \neq *$ and that there is $s \in \mathsf{exp}(l')$ for which $l \sqcup_{\mathcal{L}} s = l$. Intuitively, l and l' must carry some piece of information in common in order to $\mathtt{match}$. We sometimes refer to $\mathtt{match}$ as the *closure predicate*.

Example 25. *Let X be a set and consider the node labels algebras of the following form:*

1. $\mathcal{L}^X_{\mathsf{1D}} := \langle \mathsf{Pow}(X), X\backslash\cdot, \cup, \cap, X, \varnothing \rangle$, *where the underlying partial order is* $\subseteq$.
2. $\mathcal{L}^X_{\mathsf{2D}} := \left\langle \mathsf{Pow}(X)^2, \cdot^{\mathsf{c}\mathcal{L}^{\mathsf{X}}_{\mathsf{2D}}}, \sqcup_{\mathcal{L}^X_{\mathsf{2D}}}, \sqcap_{\mathcal{L}^X_{\mathsf{2D}}}, (X,X), (\varnothing,\varnothing) \right\rangle$, *with* $(Y,Z)^{\mathsf{c}\mathcal{L}^{\mathsf{X}}_{\mathsf{2D}}} := (X\backslash Y, X\backslash Z)$, *and with* $(Y,Z) \sqcup_{\mathcal{L}^X_{\mathsf{2D}}} (Y',Z') := (Y \cup Y', Z \cup Z')$ *and* $(Y,Z) \sqcap_{\mathcal{L}^X_{\mathsf{2D}}} (Y',Z') := (Y \cap Y', Z \cap Z')$ *, for all* $Y, Y', Z, Z' \subseteq X$. *The underlying partial order, which we also denote by* $\subseteq$, *is such that* $(Y,Z) \subseteq (Y',Z')$ *if, and only if,* $Y \subseteq Y'$ *and* $Z \subseteq Z'$.

We will usually take X to be $L_\Sigma(P)$, in which case we omit the superscript in the notation above, thus simply writing $\mathcal{L}_{\mathsf{1D}}$ and $\mathcal{L}_{\mathsf{2D}}$. In this way, in the first case, the information carried out by a tree is a set of formulas, intended to represent the formulas derived when we build one-dimensional proofs. The second definition is similar, but will be employed in two-dimensional derivations, in which the information is a collection of formulas for each one of the two dimensions, represented by a pair of sets of formulas. Regarding the *closure predicates, we have the following for the above examples, respectively:*

1. $Y \,\texttt{match}\, Z$ *if, and only if,* $Y \cap Z \neq \varnothing$.
2. $(Y, Z) \,\texttt{match}\, (Y', Z')$ *if, and only if,* $Y \cap Y' \neq \varnothing$ *or* $Z \cap Z' \neq \varnothing$.

In what follows, we omit curly braces to simplify the notation. Here are some examples of expansors over $\mathcal{L}_{\mathsf{1D}}$*:*

$$\frac{\varphi \vee \psi}{\varphi, \psi} \quad \frac{\varphi \to \psi, \varphi}{\psi} \quad \frac{\varphi, \neg\varphi}{\varnothing}$$

And here are some examples of expansors over $\mathcal{L}_{\mathsf{2D}}$*:*

$$\frac{(\{\varphi \wedge \psi\}, \varnothing)}{(\{\varphi\}, \varnothing)} \quad \frac{(\{\varphi\}, \{\psi\})}{(\varnothing, \{\varphi \to \psi\})} \quad \frac{(\{\varphi \to \psi\}, \varnothing)}{(\{\varphi\}, \{\psi\})} \quad \frac{(\{\varphi\}, \{\varphi\})}{(\varnothing, \varnothing)}$$

Finally, we define the expansion functions for each of the above examples, respectively, as:

- $\mathsf{exp}(l) := \begin{cases} * & \text{if } l = \varnothing \\ \{\{x\} \mid x \in l\} & \text{otherwise} \end{cases}$
- $\mathsf{exp}(l) := \begin{cases} * & \text{if } l = (\varnothing, \varnothing) \\ \{(y, \varnothing) \mid y \in Y\} \cup \{(\varnothing, z) \mid z \in Z\} & \text{otherwise, with } l = (Y, Z) \end{cases}$

We proceed now by defining the structures called *rooted trees*, which at first will not carry any information. We will then associate labels from $\mathcal{L}$ to the nodes of these trees (yielding *labelled rooted trees*). This will allow us to talk about deriving a piece of information from another given piece of information via the construction of an appropriate rooted labelled tree using the available expansors.

Definition 26. *A* rooted tree t *is a poset* $\langle \mathsf{nds}(t), \preceq^t \rangle$ *such that, for all* nodes $n \in \mathsf{nds}(t)$, *the set* $\mathsf{actrs}^t(n) := \{n' \in \mathsf{nds}(t) \mid n' \prec^t n\}$ *of* ancestors *of* n *is well-ordered under* $\prec^t$ *and there is a single minimal element of* $\preceq^t$*, called the* root of t *and denoted by* $\mathsf{rt}(t)$*. A* branch *of* t *is a maximal chain of* $\preceq^t$*. When every branch of* t *has a maximal element,* t

is said to be bounded. *Such maximal elements are called the* leaves *of* t. *See Figure 3.1 for some illustrations.*

Given $n \in \mathsf{nds}(t)$, the *height* of n in t is defined as the order type of $\mathsf{actrs}^t(n)$, that is, the ordinal (see [38] for details on the theory of ordinal numbers) isomorphic to $\mathsf{actrs}^t(n)$. The *height* of t itself is the least ordinal greater than the height of each of the nodes of t. The *length* of a branch of t is its order type, being denoted by $\mathsf{len}^t(b)$. Given an ordinal α, the set of all nodes of t of height α is the α*-level of* t. The supremum of the cardinalities of the levels of t is the *width* of t. We will not impose any constraints on the height and on the width of rooted trees, so they can grow infinitely in both dimensions. Moreover, the immediate predecessor of n, when it exists, is called the *parent* of n; the set of *descendants* of n is $\mathsf{dcts}^t(n) := \{n' \in \mathsf{nds}(t) \mid n \leq^t n'\}$, the immediate descentants of n are the *children* of n, and the set $\mathsf{sibl}^t(n)$ of *siblings* of n is the set $\{n' \in \mathsf{nds}(t) \mid \mathsf{actrs}^t(n') = \mathsf{actrs}^t(n)\}$. Note that we consider a node to be a sibling of itself. In finite trees, the set of siblings is the same as the set of nodes that share the same parent with n. On the other hand, when we allow the set of ancestors of n to be infinite, there may be no parent node of n, so we need to consider the whole set of ancestors of n to identify its siblings. We may also refer to $\mathsf{actrs}^t(n)$ as the *path to* n *in* t.

Definition 27. *A rooted tree* t *together with a mapping* $\ell^t : \mathsf{nds}(t) \to L \cup \{*\}$ *is said to be* labelled by ℓ^t *or a* labelled rooted tree *for short.*

Definition 28. *Given* $n \in \mathsf{nds}(t)$, *we let* $\mathsf{sub}^t(n) := \left\langle \mathsf{dcts}^t(n), \leq^t \cap \left(\mathsf{dcts}^t(n) \times \mathsf{dcts}^t(n)\right)\right\rangle$ *be the* subtree[1] of t rooted at n. *When* t *is labelled, we let* $\ell^{\mathsf{sub}^t(n)}$ *be the restriction of* ℓ^t *to* $\mathsf{dcts}^t(n)$.

It will be useful for us sometimes to replace a subtree in a given tree t by another one (see Figure 3.3 for an illustration). When the subtree to be replaced has only the root

[1]We consider here a subtree as an upward closed set, as opposed to some definitions in the set-theoretic literature, where the dual notion — using downward closed sets — is employed.

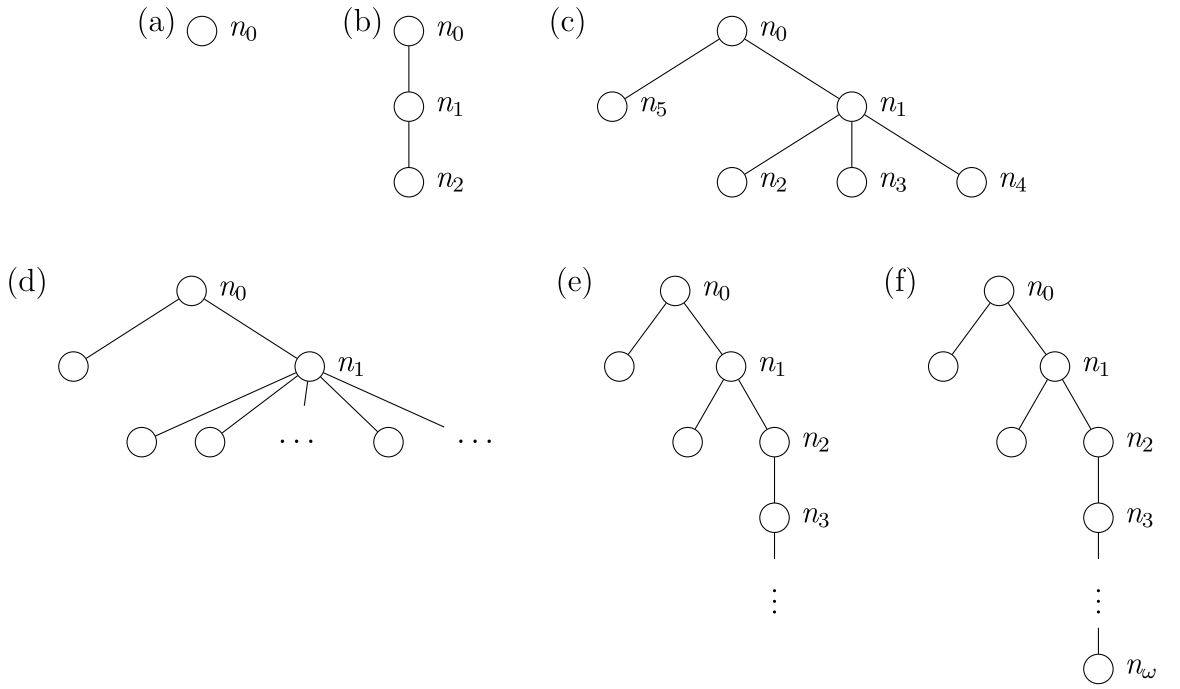


Figure 3.1.: Examples of rooted trees, all with root n_0. In (a), we see a tree with a single node, whose height is 0. In (b), we see a tree with three nodes and a single branch. Its height is 2 and n_2 is the only leaf node. In (c), we see again a tree with height 2, but now having four branches. The leaves in (c) are n_2, n_3, n_4 and n_5. The three latter trees are finite and bounded. The trees (d), (e) and (f) are infinite. In the case of (d), the node n_1 has infinitely many children (equivalently, each of its children has infinitely many siblings), and every branch is finite, thus the tree is bounded. The tree, in this case, has infinite width. In (e), there is an infinite branch, and the tree is unbounded. The height of each of its nodes is finite, but the height of the tree is ω. Finally, in (f), we have a bounded tree having a branch with infinitely many nodes. The height of n_ω is ω, while the height of the tree is $\omega + 1$.

node, this operation can be seen as a way of expanding t by "gluing" another tree at the end of one of its branches. This construction will be specially useful for Proposition 39 on page 52.

Definition 29. *Let t, t' be labelled rooted trees whose sets of nodes are disjoint, and n be a node of t. We denote by $t(n, t')$ the labelled rooted tree resulting from t by replacing the subtree $\mathsf{sub}^t(n)$ with t'.*

We may also need to delete from a tree all the subtrees rooted at the siblings of

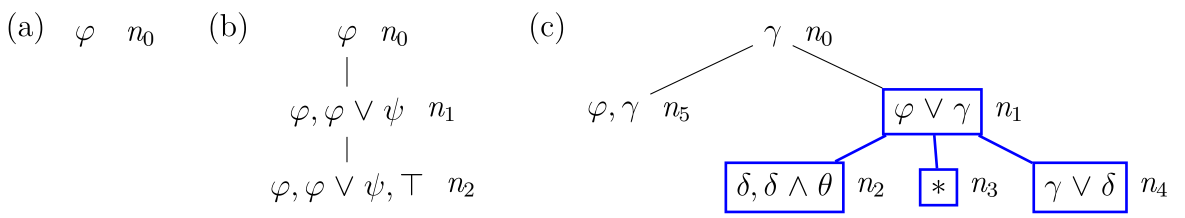


Figure 3.2.: Here are labelled versions of the trees (a), (b) and (c) in Figure 3.1. The node labels algebra is assumed to be $\mathcal{L}_{\mathsf{1D}}$, so that the labels are either sets of formulas (we omit curly braces to simplify the notation) or the discontinuation symbol. In (c), the subtree rooted at n_1 is highlighted in blue.

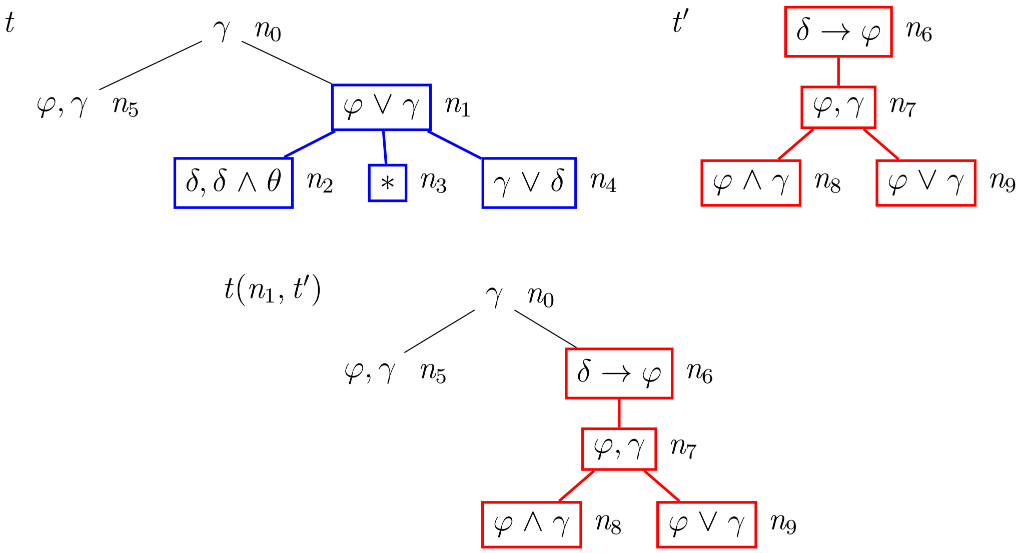


Figure 3.3.: Replacement of a subtree of t rooted at n_1 (in blue) by another tree t' (in red), yielding a new tree $t(n_1, t')$.

a given node, as described below.

Definition 30. *Given a labelled rooted tree t and a node n of t, we denote by $t \ominus n$ the labelled rooted tree resulting from t by deleting the subtrees $\mathsf{sub}^t(n')$ for all $n' \in \mathsf{sibl}^t(n)$*[2]*. See Figure 3.4 for an illustration.*

Definition 31. *Let t be a labelled rooted tree, b be a branch of t and $\{t_i\}_{i \in I}$ be a family of labelled rooted trees whose sets of nodes are pairwise disjoint and also disjoint with respect to the set of nodes of t. Denote by $t(b, \{t_i\}_{i \in I})$ the tree whose set of nodes is*

[2]Recall that this includes $\mathsf{sub}^t(n)$.

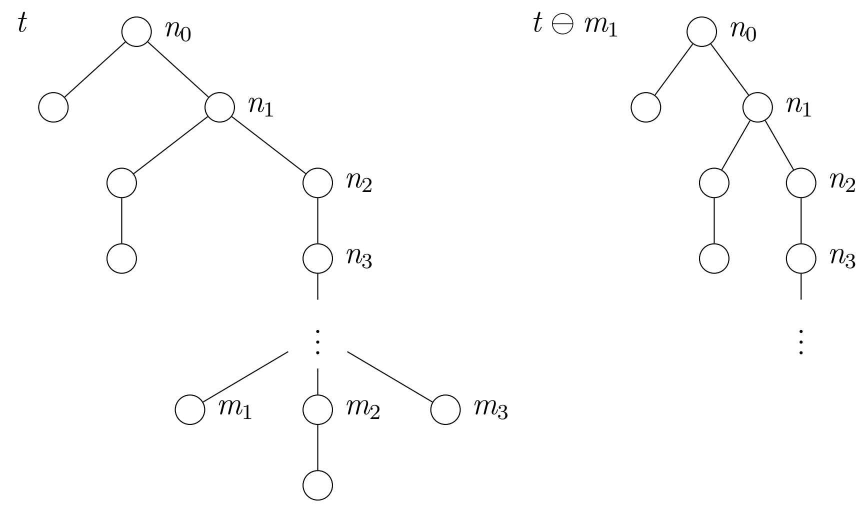


Figure 3.4.: Here, nodes m_1, m_2 and m_3 have height ω. Notice that this operation resulted in a tree which is not bounded, even though t was.

$\mathsf{nds}(t) \cup \bigcup_{i \in I} \mathsf{nds}(t_i)$, *the underlying order is as in* t *but with the nodes of* $\bigcup_{i \in I} \mathsf{nds}(t_i)$ *all descending from the notes in* b*, and the labelling is the juxtaposition of the labelling of the trees* t *and* t_i *for each* $i \in I$ *(which constitutes a function because the sets of nodes are disjoint).*

In what follows, let $\mathsf{lim}^t(n) = \bigsqcup\{\ell^t(n') \mid n' \in \mathsf{actrs}^t(n)\}$, which exists since the poset associated to $\mathcal{L}$ is complete. We are interested now in the situations where we may derive a certain informational content (represented by a label) from another given one, by means of the available expansors. We first provide the definition of a *derivation in E*, which basically describes the labelled rooted trees built by the expansors in E, and then define what a *proof in E* means.

Definition 32. *A* derivation *in a collection* E *of expansors is a bounded labelled rooted tree* t *whose root is not labelled with* $*$ *and in which, for every nonroot node* n *of* t*, there is an expansor* $(l, l') \in E$ *such that* $l \leq^{\mathcal{L}} \mathsf{lim}^t(n)$ *and either*

- $\mathsf{exp}(l') = *$, $\mathsf{sibl}^t(n) = \{n\}$, $\ell^t(n) = *$ *and* n *is a leaf; or*
- $\mathsf{exp}(l') = S \neq *$, $|\mathsf{sibl}^t(n)| = |S|$*, and, to each* $m \in S$*, there corresponds a node* $n' \in \mathsf{sibl}^t(n)$ *such that* $\ell^t(n') = \mathsf{lim}^t(n) \sqcup_{\mathcal{L}} m$*. We say that such expansor was* applied

to $\mathsf{actrs}^t(n)$*,* expanding *it and* producing $\mathsf{sibl}^t(n)$*. If the produced nodes have a parent node, we say, more simply, that e* expands *such node.*

The set of all derivations in E (over $\mathcal{L}$) is denoted by $\mathsf{Der}(\mathcal{L}, E)$*.*

In a derivation t, then, every nonroot node n has its presence justified by an application of an appropriate expansor to the path to n in t. This application produces not only n, but all of its siblings. Equivalently, it produces new branches, one for each sibling of n. The amount of such branches, as well as the labels of the produced siblings, are determined by the expansion function exp and the succedent of the applied expansor. Nodes labelled with $*$ are always leaves; the branches in which they are located are said to be *discontinued*. We may call these very nodes *discontinued* too. To each branch in a derivation there corresponds a chain of expansors that justify the presence of each node in the branch. We say that each of these expansors were *applied in the branch*.

Definition 33. *Let $l, l' \in L$. A* proof *t of (l, l') in E is a derivation in E such that*

- $\ell^t(\mathsf{rt}(t)) \leq^{\mathcal{L}} l$*; and*
- *for every leaf n of t, we have either $\ell^t(n) = *$ or $\ell^t(n)\,\mathtt{match}\,l'$.*

We write $l \vdash_E l'$ whenever there is a proof in E of (l, l'), and say that (l, l') is derivable in *E.*

Note that, given a proof t of (l, l') in E and a nonleaf node n of t produced by an expansor e with succedent m, the subtree of t rooted at n is a proof of $(\mathsf{lim}^t(n) \sqcup_{\mathcal{L}} s, l')$, for a label $s \in \mathsf{exp}(m)$.

An application of an expansor may produce new branches with no new information. When this happens in a proof, we will show that such an application can be avoided without harm. One of the consequences of this fact is that we will never need to apply an expansor twice on the same branch in order to provide a proof in E.

Definition 34. *Let t be a proof of (l, l') in E. If, for some nonleaf node n,* $\mathsf{sibl}^t(n)$ *was produced by an expansor $e \in E$ such that $\mathsf{lim}^t(n) = \ell^t(n)$, then this application of e is* *said to be* irrelevant. *A proof without irrelevant applications of expansors is said to be* concise.

Proposition 35. *$l \vdash_E l'$ if, and only if, there is a concise proof of (l, l') in E.*

Proof. Let t be a proof of (l, l') in E. The right-to-left direction is obvious. For the converse, we prove that every irrelevant application of an expansor in t may be removed. Suppose that a nonleaf node n of t was produced by an application of $e := (m, m') \in E$ and that (a): $\mathsf{lim}^t(n) = \ell^t(n)$. Since n is a nonleaf node, its children were produced by an application of an expansor $e' \in E$. But e' could have been used instead of e, given (a). Since t is a proof of (l, l') in E , replacing the application of e by the application of e' results in a tree which is still a proof of (l, l') in E. By repeating this process for each irrelevant application of expansors in t, we will end up with a concise proof of (l, l') in E. $\square$

Corollary 36. *It is not necessary to apply the same expansor twice in a branch of a proof.*

Proof. A second application of an expansor in a branch could either produce at least one nonleaf node or only leaf nodes. In the first case, this would be an irrelevant application, and thus could be removed in view of Proposition 35. If only leaf nodes were produced, then we could have used only the first application of the expansor without any loss. $\square$

Whenever we show that (l, l') is provable in E, we are able to use (l, l') as an expansor without incurring the risk of proving more than what we could prove before. Proposition 39 below formalizes and prove this observation, which will be useful specially in proving Proposition 43 on page 54.

Definition 37. *An expansor (l, l') is said to be* derivable *in E provided $l \vdash_E l'$.*

Definition 38. *Given a labelled rooted tree t and a label l, let $t \oplus l$ be the tree that differs from t only in the labelling: $\ell^{t \oplus l}(n) := \ell^t(n) \sqcup_{\mathcal{L}} l$, for all $n \in \mathsf{nds}(t)$.*

Proposition 39. *If the expansor $e = (l, l')$ is derivable in E, then* $\vdash_E = \vdash_{E\cup\{e\}}$.

Proof. The left-to-right direction is obvious. For the converse, suppose that $m \vdash_{E\cup\{e\}} m'$, witnessed by a derivation t, which, without loss of generality, we assume to be concise. Let t_e be a concise proof of e in E, and assume without loss of generality that it has more than one node, that the label of its root is l and that $\mathsf{nds}(t) \cap \mathsf{nds}(t_e) = \varnothing$. We want to show that applications of e in t may be replaced by applications of expansors in E.

Assume that (a): e was applied to $\mathsf{actrs}^t(n)$. Then (b): $l \leq^{\mathcal{L}} \mathsf{lim}^t(n)$, and, for each $s \in \mathsf{exp}(l')$, there is a derivation t^e_s witnessing that $\mathsf{lim}^t(n) \sqcup_{\mathcal{L}} s \vdash_{E\cup\{e\}} m'$. Since t is concise, we have that such derivations actually bear witness to $\mathsf{lim}^t(n) \sqcup_{\mathcal{L}} s \vdash_E m'$.

Let $e' = (s, s')$ be the expansor applied to the root node of t_e. Then $s \leq^{\mathcal{L}} l$, and, for each $k \in \mathsf{exp}(s')$, a node is produced with label $l \sqcup_{\mathcal{L}} k$, being the root of a subtree t_k of t_e that is a proof of $l \sqcup_{\mathcal{L}} k \vdash_E l'$. Consider the tree $t^*_k := t_k \oplus \mathsf{lim}^t(n)$ (recall Definition 38). In this way, the root of each t^*_k is $\mathsf{lim}^t(n) \sqcup_{\mathcal{L}} k$ (in view of (b)). For each $k \in \mathsf{exp}(s')$, then, t^*_k is a proof of $\mathsf{lim}^t(n) \sqcup_{\mathcal{L}} k \vdash_E l'$.

Let $t' := t \ominus n$ (recall Definition 30), and $t'' := t'(\mathsf{actrs}^t(n), \{t^*_k\}_{k\in\mathsf{exp}(s')})$ (recall Definition 31). The resulting tree is a derivation in E, but still not a proof of (m, m'). The new leaf nodes n' not labelled with $*$ possibly present in t'' are such that $\ell^{t'}(n')$ $\mathtt{match}$ l'. Thus, there is $r \in \mathsf{exp}(l')$, such that $\ell^{t'}(n') \sqcup_{\mathcal{L}} r = \ell^{t'}(n')$. Consider the tree $t^e_r \oplus \ell^{t'}(n')$, and note that its root has label $\ell^{t'}(n')$ — since $\mathsf{lim}^t(n) \leq^{\mathcal{L}} \ell^{t'}(n')$. Thus we may use it instead of the leaf n', via the operation described in Definition 29. Doing this for every such leaf n', we finish the proof. ☐

We work now on a process for building, or searching for, a proof in E of a given (l', l). It basically consists in selecting and expanding the branches of a given tree via the available expansors, until the desired derivation is achieved or no (relevant) application of an expansor is possible.

Definition 40. *Let t be a labelled rooted tree and b be a branch of t. The* label *of b,* *denoted by* $\ell^t(b)$*, is defined as* $\bigsqcup\{\ell^t(n) \mid n \in b\}$*. We say that an expansor* $(l, l') \in E$ relevantly applies to b *in case*

1. $l \leq^{\mathcal{L}} \ell^t(b)$*; and*
2. $\ell^t(b) <^{\mathcal{L}} m \sqcup_{\mathcal{L}} \ell^t(b)$*, for each* $m \in \mathsf{exp}(l')$*, in case* $\mathsf{exp}(l') \neq *$.

When there is such an expansor, b *is said to be* amenable to a relevant expansion in E*.*

Definition 41. *Let* t *be a rooted labelled tree,* b *be a branch of* t *and* $e := (l, l') \in E$ *be an expansor that relevantly applies to* b*. Set* $S := \mathsf{exp}(l')$ *and define* $\Delta(S)$ *to be a set of nodes not in* $\mathsf{nds}(t)$ *such that*

$$\Delta(S) := \begin{cases} \{n_m \mid m \in S\}, & \text{if } S \neq * \\ \{n_*\}, & \text{otherwise} \end{cases}$$

We define the expansion of t on b by e *as the rooted labelled tree* $t\downarrow_e^b$ *such that:*

- $\mathsf{nds}(t\downarrow_e^b) := \mathsf{nds}(t) \cup \Delta(S)$
- $\preceq^{t\downarrow_e^b} := \preceq^t \cup \{(n', n) \mid n \in \Delta(S), n' \in b\}$
- $\ell^{t\downarrow_e^b}(n) := \ell^t(n)$ *for all* $n \in \mathsf{nds}(t)$
- $\ell^{t\downarrow_e^b}(n_s) := \ell^t(b) \sqcup_{\mathcal{L}} s$ *for all* $n_s \in \Delta(S)$*, if* $\Delta(S) \neq \{n_*\}$
- $\ell^{t\downarrow_e^b}(n_s) := *$ *if* $\Delta(S) = \{n_*\}$

Given a labelled rooted tree t, we fix an order on the collection B_t of its branches amenable to a relevant expansion. Also, given a branch b of t, we fix an order on the set of expansors relevantly applicable to b, and let E_b be a partial function that is undefined if no expansor is relevantly applicable to b or produces the first expansor applicable to b according to the fixed order. Let $T(n, l)$ be the set of all derivations whose root is n labelled with l. The tree in this set with a single node is denoted by $t_\perp$. Moreover, whenever $g : B \to B$ is a function on a complete partially ordered set B, we let, for all $x \in B$, $g^0(x) := x$, $g^{\alpha+1}(x) := g(g^\alpha(x))$ for sucessor ordinals $\alpha + 1$, and

$g^\lambda(x) = \bigsqcup_{\alpha<\lambda} g^\alpha(x)$ for limit ordinals λ.

Definition 42. *Given $l, l' \in L$, we define the mapping $f_{l,l'}$ on $T(n,l)$ such that:*

- $f_{l,l'}(t) = t$, *if t is a proof of (l, l') in E or, for each branch $b \in B_t$, E_b is undefined; otherwise,*
- $f_{l,l'}(t) = t{\downarrow}^b_e$, *where b is the first branch of t amenable to a relevant expansion and e is E_b.*

Also, let $T_{l'}(n,l) := \{t \in T(n,l) \mid t = f^\alpha_{l,l'}(t_\bot),$ for some ordinal $\alpha\}$ — namely, the set of all derivations achievable from $t_\bot$ via consecutive applications of $f_{l,l'}$.

Notice that the relation $\leq_{l'}$ on $T_{l'}(n,l)$ such that $t_1 \leq_{l'} t_2$ iff $t_2 = f^\alpha_{l,l'}(t_1)$ for some ordinal α is a complete partial order with least element $t_\bot$. The supremum of a chain Y over this order is the tree $\bigsqcup Y := \langle \bigcup_{t\in Y} \mathsf{nds}(t), \bigcup_{t\in Y} \leq^t \rangle$ labelled by $\bigcup_{t\in Y} \ell^t$. Moreover, $f_{l,l'}$ is monotonic with respect to $\leq_{l'}$.

We work now to prove some properties satisfied by the derivability relation $\vdash_E$. The reader will notice that, provided appropriate instantiations of the node labels algebra ($\mathcal{L}_{\mathsf{1D}}$ and $\mathcal{L}_{\mathsf{2D}}$, respectively), they will correspond to properties of Set-Set consequence relations and B-consequence relations.

Proposition 43. *For all $l, l', m, m' \in L$,*

(Ot) $l \vdash_E l'$ *whenever* $l\,\mathtt{match}\,l'$.

(Dt) *if* $l \vdash_E l'$, *then* $l \sqcup_{\mathcal{L}} m \vdash_E l' \sqcup_{\mathcal{L}} m'$

(Ct) *if for all $s \in L$ such that $l \leq^{\mathcal{L}} s$ and $l' \leq^{\mathcal{L}} s^{\mathsf{c}\mathcal{L}}$ we have $l \sqcup_{\mathcal{L}} s \vdash_E l' \sqcup_{\mathcal{L}} s^{\mathsf{c}\mathcal{L}}$, then $l \vdash_E l'$.*

Moreover, $\vdash_E$ is the least relation containing all expansors in E and satisfying the above properties.

Proof. For **(Ot)**, assume that $l\,\mathtt{match}\,l'$ and consider a tree with a single node labelled with $l^* := l \sqcap_{\mathcal{L}} l'$. Then clearly $l^* \leq^{\mathcal{L}} l$ and $l^*\,\mathtt{match}\,l'$. For **(Dt)**, suppose that t bears witness to $l \vdash_E l'$. Then t itself bears witness to $l \sqcup_{\mathcal{L}} m \vdash_E l' \sqcup_{\mathcal{L}} m'$, since $\ell^t(\mathsf{rt}(t)) \leq^t l$, $l \leq^t l \sqcup_{\mathcal{L}} m$, and $\leq^t$ is transitive; in addition to the fact that `match` is monotonic on the second argument with respect to the order $\leq^{\mathcal{L}}$.

For **(Ct)**, we first assume without loss of generality that we do not have l `match` l'. Suppose that for all $s \in L$ such that $l \leq^{\mathcal{L}} s$ and $l' \leq^{\mathcal{L}} s^{\mathsf{c}\mathcal{L}}$ we have $l \sqcup_{\mathcal{L}} s \vdash_E l' \sqcup_{\mathcal{L}} s^{\mathsf{c}\mathcal{L}}$. Recall, in view of Proposition 39, that (a): for all $s \in L$ such that $l \leq^{\mathcal{L}} s$ and $l' \leq^{\mathcal{L}} s^{\mathsf{c}\mathcal{L}}$, we are allowed to use $(s, s^{\mathsf{c}\mathcal{L}})$ as an expansor in derivations. Since $f_{l,l'}$ is monotonic with respect to $\leq_{l'}$, and since $\leq_{l'}$ is complete, $f_{l,l'}$ has a least fixed point, given by $f^{\alpha}_{l,l'}(t_{\perp})$, for some ordinal α. By the definition of $f_{l,l'}$, we have two possibilities: either this tree is a proof of (l, l') or some of its branches are not amenable to a relevant expansion. The latter, however, cannot happen, in view of (a), and we are done.

We prove now that the derivability relation is the least relation satisfying those three properties. Given a concise derivation t, let $l_t := \ell^t(\mathsf{rt}(t))$ and, for every leaf n of a nondiscontinued branch b of t, let m_n be $\ell^t(n) \backslash \mathsf{lim}^t(n)$ (which must be an atom, given the definition of exp), and l'_t be the supremum of all m_n. Then t is a proof of (l_t, l'_t) in E. It turns out that, for all $l, l' \in L$, $l \vdash_E l'$ if, and only if, $l_t \leq^t l$ and $l'_t \leq^t l'$ for some derivation t in E. The right to left direction is easy, given **(Dt)**. For the converse direction, let t be a concise proof of (l, l') in E. Then $l_t = \ell^t(\mathsf{rt}(t)) \leq^t l$. Also, let n be a leaf node of t not labelled with $*$, and let $s_n := \ell^t(n) \sqcap_{\mathcal{L}} l' \neq \perp_{\mathcal{L}}$. Assume without loss of generality that $\mathsf{lim}^t(n) \sqcap_{\mathcal{L}} l' = \perp_{\mathcal{L}}$. Then $s_n = (\mathsf{lim}^t(n) \sqcup_{\mathcal{L}} m_n) \sqcap_{\mathcal{L}} l' = (\mathsf{lim}^t(n) \sqcap_{\mathcal{L}} l') \sqcup_{\mathcal{L}} (m_n \sqcap_{\mathcal{L}} l') \neq \perp_{\mathcal{L}}$, thus $s_n = m_n \sqcap_{\mathcal{L}} l'$, and $s_n \leq^t m_n$, but m_n is an atom, so $s_n = m_n$. Therefore, $m_n \leq^t l'$, for each n, thus $l'_t \leq^t l'$, as desired. In this way, if $l \vdash_E l'$, and we prove that $l_t \vdash l'_t$, for a derivation t with $l_t \leq l$ and $l'_t \leq l'$, we obtain $l \vdash_E l'$ by **(Dt)**. In other words, we just need to prove, for all derivations t, that $l_t \vdash_E l'_t$.

Given a derivation t, a branch b of t and an ordinal h, let $t[b, h]$ denote the tree resulting from t by removing all descendants of the node n_h of height h in b, except for the node n_h itself. Note that, since t is bounded, so is $t[b,h]$. Let $\vdash$ be a binary relation on L containing the expansors in E and satisfying **(Ot)**, **(Dt)** and **(Ct)**. Assume that $l_t \vdash_E l_{t'}$. We proceed to prove by induction on $0 \leq h \leq \mathsf{len}^t(b)$ that $P(h)$: $l_{t[b,h]} \vdash l'_{t[b,h]}$. In particular, when $h = \mathsf{len}^t(b)$, $t[b,h] = t$, and we will have $l_t \vdash l'_t$, as desired. For $h = 0$, $t[b,0]$ has only the root of t, so $l_{t[b,0]} = l'_{t[b,0]}$. Thus, by **(Ot)** we have $l_{t[b,0]} \vdash l'_{t[b,0]}$. Assume that (IH): $P(j)$ holds for all $1 \leq j < i$, with $i \geq 1$, and that the expansor (k,k') produced n_i (a node with height i in t). Note that $k' \leq^{\mathcal{L}} l'_{t[b,i]}$. Our intention now is to use **(Ct)** to prove $l_t = l_{t[b,i]} \vdash l'_{t[b,i]}$. Let $s \in L$ and suppose that $l_{t[b,i]} \leq^{\mathcal{L}} s$ and $l'_{t[b,i]} \leq^{\mathcal{L}} s^{\mathsf{c}\mathcal{L}}$. We consider two cases:

1. there is a node n_u $(u < i)$ in $\mathsf{actrs}^t(n_i)$ such that (a): $m_u \leq^{\mathcal{L}} s^{\mathsf{c}\mathcal{L}}$. We have then, by (IH), $l_{t[b,u]} \vdash l'_{t[b,u]}$. Notice that all the other leaf nodes descending from n_u were removed, but the other leaf nodes are still in $t[b,u]$. In this way, $l'_{t[b,u]} \backslash m_u \leq^{\mathcal{L}} l'_{t[b,i]} \leq^{\mathcal{L}} s^{\mathsf{c}\mathcal{L}}$. Then $l'_{t[b,u]} = l'_{t[b,u]} \backslash m_u \sqcup_{\mathcal{L}} m_u \leq^{\mathcal{L}} s^{\mathsf{c}\mathcal{L}} \sqcup_{\mathcal{L}} m_u = s^{\mathsf{c}\mathcal{L}}$ (by (a) and the fact that $m_u \leq^{\mathcal{L}} l_{t[b,u]}$). Since $l_{t[b,u]} = l_t \leq^{\mathcal{L}} s$, by (b) we get $s \vdash s^{\mathsf{c}\mathcal{L}}$.
2. for all n_u in $\mathsf{actrs}^t(n_i)$, $m_u \leq^{\mathcal{L}} s$. Then $l_t \sqcup_{\mathcal{L}} \bigsqcup_{n_u \in \mathsf{actrs}^t(n_i)} m_u = \mathsf{lim}^t(n_i) \leq^{\mathcal{L}} s$. But $k \leq^{\mathcal{L}} \mathsf{lim}^t(n_i)$, thus $k \leq^{\mathcal{L}} s$. As clearly $k' \leq s^{\mathsf{c}\mathcal{L}}$, and $k \vdash k'$, we have by **(Dt)** $s \vdash s^{\mathsf{c}\mathcal{L}}$, as desired.

Therefore, for all $s \in L$ such that $l_t = l_{t[b,i]} \leq^{\mathcal{L}} s$ and $l'_{t[b,i]} \leq^{\mathcal{L}} s^{\mathsf{c}\mathcal{L}}$, we have $s \vdash s^{\mathsf{c}\mathcal{L}}$. Then, by **(Ct)**, we have $l_{t[b,i]} \vdash l'_{t[b,i]}$, and we are done. $\square$

3.2. Symmetrical H-systems for one-dimensional consequence relations

The Set-Set H-formalism was first developed by [57] and applied recently by [17, 43] in the axiomatization of logical matrices, including those that are partial non-deterministic. In contrast to the Set-Fmla H-formalism explained above, rules have sets as succedents, instead of a single formula, and derivations are bounded rooted trees labelled with sets of formulas instead of being just sequences of formulas. In other words, using the terminology of Section 3.1, derivations are bounded labelled rooted trees over the node labels algebra $\mathcal{L}_{\mathsf{1D}}$, as described in Example 25. Most of what comes in the remainder of this section, in fact, introduces names that are closer to the ones found in the literature in Set-Set calculi for the objects and notions introduced in Section 3.1. Consequently, all results proved there hold in this particular context.

Definition 44. *A* Set-Set rule of inference R *is a collection of* Set-Set *statements* (Γ, Δ)*, called the* rule instances of R*, where* Γ *is the* antecedent *and* Δ *is the* succedent *of the rule instance. A* Set-Set system R *is a collection of* Set-Set *rules of inference. We denote by* $\mathsf{Inst}(\mathsf{R})$ *the union of all rules of inference of* R*, that is, all the rule instances of* R*. A* Set-Set *rule of inference is* schematic *when it is the collection of all substitution instances of a representative* Set-Set *statement* s*, the* schema *of that very rule. We denote such a rule by* R_{s}*. A* Set-Set *system is* schematic *when all of its rules of inference are schematic.*

Clearly, Set-Set rule instances correspond to expansors over $\mathcal{L}_{\mathsf{1D}}$, as defined in Section 3.1. Schematic Set-Set systems are usually specified just by listing the rule schemas that induce their rules of inference, as we exemplify below.

Example 45. *Consider the schematic system* $\mathsf{R}_S^{\to\neg}$ *given by the following rule schemas:*

$$\frac{\neg p, q}{p \to q}\mathsf{s}_1^S \qquad \frac{p, \neg q}{\neg(p \to q)}\mathsf{s}_2^S \qquad \frac{p \to q, \neg q}{\neg p}\mathsf{s}_3^S$$
$$\frac{\neg(p \to q)}{p}\mathsf{s}_4^S \qquad \frac{\neg(p \to q)}{\neg q}\mathsf{s}_5^S \qquad \frac{p, p \to q}{q}\mathsf{s}_6^S$$
$$\frac{p}{\neg\neg p}\mathsf{s}_7^S \qquad \frac{\neg\neg p}{p}\mathsf{s}_8^S \qquad \frac{}{p, \neg p}\mathsf{s}_9^S$$

Notice that what distinguishes this calculus from a standard Hilbert calculus is the rule of inference given by the last rule schema, whose succedent is a set with two formulas. By *the procedure described in [43], we have that the* SET-SET *consequence-relation induced by this calculus is the* $\{\to, \neg\}$*-fragment of Sobociński's three-valued logic* **S3** [60].

Example 46. *Consider the calculus* $\mathsf{R}_{BK}^{\wedge\neg}$ *consisting of the following rules of inference:*

$$\frac{p, q}{p \wedge q}\mathsf{s}_1^{BK} \quad \frac{p \wedge q}{p}\mathsf{s}_2^{BK} \quad \frac{p \wedge q}{q}\mathsf{s}_3^{BK} \quad \frac{\neg p, \neg q}{\neg(p \wedge q)}\mathsf{s}_4^{BK}$$
$$\frac{\neg(p \wedge q)}{p, \neg p}\mathsf{s}_5^{BK} \quad \frac{\neg(p \wedge q)}{q, \neg q}\mathsf{s}_6^{BK} \quad \frac{p, \neg q}{\neg(p \wedge q)}\mathsf{s}_7^{BK} \quad \frac{\neg p, q}{\neg(p \wedge q)}\mathsf{s}_8^{BK}$$
$$\frac{p}{\neg\neg p}\mathsf{s}_9^{BK} \quad \frac{\neg\neg p}{p}\mathsf{s}_{10}^{BK} \quad \frac{p, \neg p}{}\mathsf{s}_{11}^{BK}$$

By [43], this calculus axiomatizes the $\{\wedge, \neg\}$*-fragment of Bochvar's three-valued logic [15], where a third value is introduced to refer to "meaningless" sentences. In the last rule, we observe that* SET-SET *rule instances may have an empty succedent.*

Example 47. *The following* SET-SET *calculus, which we call* R_{FDE}*, axiomatizes Dunn-Belnap's FDE, again as a consequence of the procedure in [43]:*

$$\frac{p \wedge q}{p}\mathsf{s}_1^{FDE} \quad \frac{p \wedge q}{q}\mathsf{s}_2^{FDE} \quad \frac{p, q}{p \wedge q}\mathsf{s}_3^{FDE} \quad \frac{p}{p \vee q}\mathsf{s}_4^{FDE} \quad \frac{q}{p \vee q}\mathsf{s}_5^{FDE} \quad \frac{p \vee q}{p, q}\mathsf{s}_6^{FDE}$$
$$\frac{\neg(p \wedge q)}{\neg p, \neg q}\mathsf{s}_7^{FDE} \quad \frac{\neg p}{\neg(p \wedge q)}\mathsf{s}_8^{FDE} \quad \frac{\neg q}{\neg(p \wedge q)}\mathsf{s}_9^{FDE}$$
$$\frac{\neg(p \vee q)}{\neg p}\mathsf{s}_{10}^{FDE} \quad \frac{\neg(p \vee q)}{\neg q}\mathsf{s}_{11}^{FDE} \quad \frac{\neg p, \neg q}{\neg(p \vee q)}\mathsf{s}_{12}^{FDE}$$
$$\frac{p}{\neg\neg p}\mathsf{s}_{13}^{FDE} \quad \frac{\neg\neg p}{p}\mathsf{s}_{14}^{FDE}$$

We give now an intuitive account of derivations and proofs in a SET-SET system, for then introducing these concepts formally. SET-SET derivations are bounded labelled rooted trees in which the presence of each nonroot node is justified by the application of a rule instance of the SET-SET system at hand. The applicability of a rule instance in a branch of a derivation is subjected to the antecedent of this very rule being included in the union of all labels of the nodes of that branch (that is, the premises of the rule must be *satisfied* by that branch). The result of such application is the expansion of the branch with new branches, one for each formula in the succedent of the applied rule instance. Each new child node is labelled with the formulas of the ancestor nodes plus one of the formulas in the succedent of the applied rule. If the succedent of the applied rule instance is empty, the branch gets discontinued (its leaf is labelled with $*$). A proof that Ψ follows from Φ is then a derivation whose root is labelled with a subset of Φ and every leaf node either has a formula in common with Ψ or is discontinued. Notably, the Set-Fmla H-formalism is the particular case of this more general setting in which the succedent of rules are singletons and, consequently, derivations are deemed to have a single branch, being thus equivalent to a sequence of formulas, as expected. As one of the goals of the present study is to generalize this formalism to the two-dimensional environment, we proceed to a more formal presentation.

Definition 48. *Let* R *be a* Set-Set *system. An* R-derivation t *is a bounded labelled rooted tree in* $\mathsf{Der}(\mathcal{L}_{\mathsf{1D}}, \mathsf{Inst}(\mathsf{R}))$ *(check Definition 32). We represent* R*-derivations graphically as per Figure 3.5.*

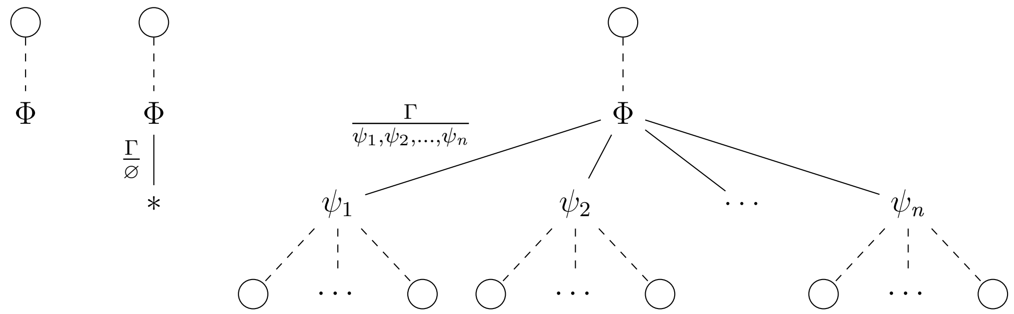


Figure 3.5.: Graphical representation of finite R–derivations, in the form of a leaf node, a discontinued node and an expanded node, respectively. The dashed edges and blank circles represent other nodes and edges that may exist in the derivation. Notice that, in the case of expanded nodes, we omit the formulas inherited from the parent node, exhibiting only the ones introduced by the applied rule of inference. Also, we have written the rule instance of the rule being applied in the second and third trees, but in practice only the name of the rule of inference, or the name of its rule schema in case it is schematic, will be written for simplicity. In these cases, we emphasize that a precondition for the application of the rule instance is that $\Gamma \subseteq \Phi$.

Definition 49. *A node* n *of an* R*-derivation* t *is called* Δ-closed *in case it is discontinued or when it is a leaf node with* $\ell^{\mathsf{t}}(\mathsf{n}) \cap \Delta \neq \varnothing$*. A branch of* t *is* Δ*-closed when it ends in a* Δ*-closed node. When every branch in* t *is* Δ*-closed, we say that* R *is itself* Δ-closed*.*

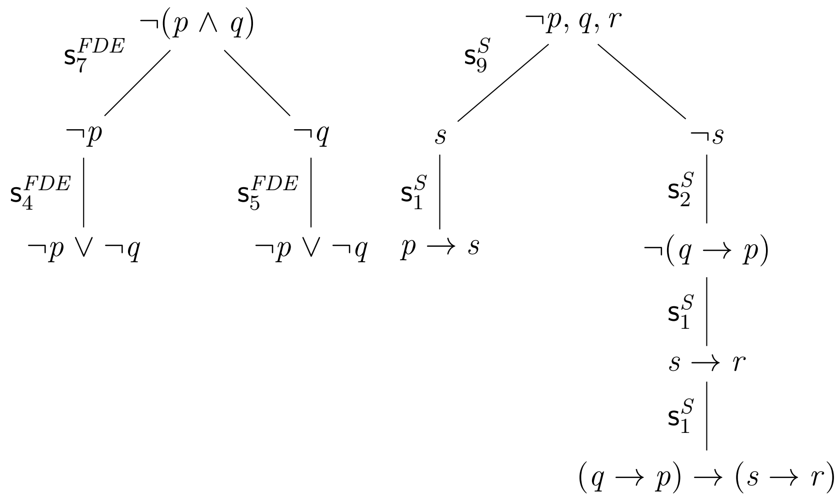
Definition 50. *An* R-proof *of a* Set-Set *statement* (Φ, Ψ) *is a* Ψ*-closed* R*-derivation* t *such that* $\ell^{\mathsf{t}}(\mathsf{rt}(\mathsf{t})) \subseteq \Phi$*.*

Consider the binary relation $\rhd_{\mathsf{R}}$ on $\mathsf{Pow}(L_\Sigma(P))$ such that $\Phi \rhd_{\mathsf{R}} \Psi$ if, and only if, there is an R-proof of (Φ, Ψ). The following result establishes a close connection between the above definition of and Set-Set consequence relations.

Proposition 51. *The relation* $\rhd_{\mathsf{R}}$ *is the smallest* Set-Set *consequence relation containing the rules of inference of* R*. If* R *is schematic, then* $\rhd_{\mathsf{R}}$ *is substitution-invariant, and, if it is finitary, then* $\rhd_{\mathsf{R}}$ *is finitary and every* R*-proof of a statement can be turned into a finite* R*-proof of the same statement.*

Proof. The relation $\rhd_{\mathsf{R}}$ is substitution-invariant because $\mathsf{Inst}(\mathsf{R})$ is closed under substitution instances. It follows directly from Proposition 43 that $\rhd_{\mathsf{R}}$ is the least substitution-invariant Set-Set consequence relation containing the rule instances of R. For the remainder of the proof, the reader is referred to the proof of Proposition 62 and Corollary 63, which are analogous results for the Set^2-Set^2 case. □

Example 52. *Below we show that* $\neg(p \wedge q) \rhd_{\mathsf{R}_{FDE}} \neg p \vee \neg q$*;* $\neg p, q, r \rhd_{\mathsf{R}_S^{\to\neg}} (q \to p) \to (s \to r), p \to s$ *and* $\neg(p \wedge q), p \rhd_{\mathsf{R}_{BK}^{\wedge\neg}} \neg q$*:*



Given a Set-Set system R and a Σ-nd-matrix $\mathbb{M}$, we say that R is *sound* for $\mathbb{M}$ when $\rhd_{\mathsf{R}} \subseteq \rhd_{\mathbb{M}}$ and that it is *complete* for $\mathbb{M}$ when the converse inclusion holds. When it is both sound and complete for $\mathbb{M}$, we say that it is *adequate* for $\mathbb{M}$ or an *axiomatization* for $\mathbb{M}$. It was proved in [43, 17] that whenever $\mathbb{M}$ satisfies a very inclusive sufficient expressiveness requirement, one can algorithmically produce a Set-Set axiomatization for it. More than that, the produced systems satisfy the property of Θ-*analyticity*, for a particular $\Theta \subseteq L_\Sigma(P)$, according to which the subformulas of the Set-Set statement being proved plus the formulas resulting from substituting those subformulas in the formulas in Θ are enough to produce the desired proof when it exists. The reader is referred to the latter references for a thorough exposition, as Set-Set systems are not the main topic of the present study. In Chapter 4, we will present in detail the property of Θ-analyticity for our generalization of the Set-Set formalism to the two-dimensional environment, which we call $\textsc{Set}^2$-$\textsc{Set}^2$ formalism.

4. Two-dimensional Hilbert-style formalism

Our goal in this chapter is to develop an H-formalism for two-dimensional logics inspired in the H-formalism for Set-Set logics detailed in Section 2.5.2, which we call *two-dimensional symmetrical formalism* or Set^2-Set^2 *H-formalism*. After presenting and illustrating the main definitions, taking advantage of the general notions and results presented in Section 3.1, we will describe an exponential proof-search algorithm over this novel deductive formalism, giving a correctness proof and performing a complexity analysis.

4.1. Rules of inference and derivations

We begin by defining what we mean by rules of inference in a Set^2-Set^2 system. As we are designing an H-formalism, rule instances are expected to be Set^2-Set^2 statements. Notice that, in Definition 53 below, we have changed the positioning of the sets of formulas in the statement representing a rule instance, in order to facilitate the development of proofs in trees growing downwards from the premises to the conclusion. We emphasize, however, that this is just a modification in notation; the objects being denoted are still the same. Moreover, we change a bit the visual so as to make it clear when we are referring to a rule instance or to an ordinary Set^2-Set^2 statement.

Definition 53. *A* $\textsc{Set}^2$-$\textsc{Set}^2$ rule of inference R *is a collection of* $\textsc{Set}^2$-$\textsc{Set}^2$ *statements* r*, called* $\textsc{Set}^2$-$\textsc{Set}^2$ rule instances *and denoted by* $\frac{\Phi_{\mathsf{Y}} \,\|\, \Phi_{\mathsf{N}}}{\Phi_{\mathsf{\lambda}} \,\|\, \Phi_{\mathsf{И}}}$*, where* $(\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}})$ *is the* antecedent *and* $(\Phi_{\mathsf{\lambda}}, \Phi_{\mathsf{И}})$ *is the* succedent *of the rule instance. We let* $\mathsf{branch}(r) := \Phi_{\mathsf{\lambda}} \cup \Phi_{\mathsf{И}}$ *be the* branching *of* r*. A* $\textsc{Set}^2$-$\textsc{Set}^2$ system *(sometimes called* $\textsc{Set}^2$-$\textsc{Set}^2$ *calculus)* $\mathfrak{R}$ *is a collection of* $\textsc{Set}^2$-$\textsc{Set}^2$ *rules of inference. We denote by* $\mathsf{Inst}(\mathfrak{R})$ *the union of all rules of inference of* $\mathfrak{R}$*, that is, all the rule instances of* $\mathfrak{R}$*.*

Definition 54. *A* $\textsc{Set}^2$-$\textsc{Set}^2$ *rule instance is* finitary *when the sets in their antecedent and succedent are finite. A* $\textsc{Set}^2$-$\textsc{Set}^2$ *rule of inference is* finitary *when all of its rule instances are finitary. Finally, a* $\textsc{Set}^2$-$\textsc{Set}^2$ *system is* finitary *when all its rules of inference are finitary.*

A rule of inference is commonly specified schematically, namely, as the collection of all substitution instances of a representative $\textsc{Set}^2$-$\textsc{Set}^2$ statement.

Definition 55. *A* $\textsc{Set}^2$-$\textsc{Set}^2$ *rule of inference is* schematic *when it is the collection of all substitution instances of a* $\textsc{Set}^2$-$\textsc{Set}^2$ *statement* $\mathfrak{s}$*, called the* rule schema *of that very rule. The rule of inference with schema* $\mathfrak{s}$ *will be denoted by* $R_{\mathfrak{s}}$*. A* $\textsc{Set}^2$-$\textsc{Set}^2$ *system is* schematic *when all of its rules of inference are schematic.*

In view of the previous definition, a schematic $\textsc{Set}^2$-$\textsc{Set}^2$ system may be presented by just indicating a collection of $\textsc{Set}^2$-$\textsc{Set}^2$ rule schemas, as exemplified below.

Example 56. *Consider the schematic* $\textsc{Set}^2$-$\textsc{Set}^2$ *calculus* $\mathfrak{R}^I$ *given by the following* *rule schemas:*

$$\frac{p \quad \parallel}{p \vee q \quad \parallel}\vee^4_1 \qquad \frac{q \quad \parallel}{p \vee q \quad \parallel}\vee^4_2 \qquad \frac{\parallel \quad p, q}{\parallel \quad p \vee q}\vee^4_3 \qquad \frac{\parallel \quad p \vee q}{\parallel \quad q}\vee^4_4 \qquad \frac{\parallel \quad p \vee q}{\parallel \quad p}\vee^4_5$$

$$\frac{p \wedge q \quad \parallel}{p \quad \parallel}\wedge^4_1 \qquad \frac{p \wedge q \quad \parallel}{q \quad \parallel}\wedge^4_2 \qquad \frac{p, q \quad \parallel}{p \wedge q \quad \parallel}\wedge^4_3 \qquad \frac{\parallel \quad q}{\parallel \quad p \wedge q}\wedge^4_4 \qquad \frac{\parallel \quad p}{\parallel \quad p \wedge q}\wedge^4_5$$

$$\frac{\parallel \quad \neg p}{p \quad \parallel}\neg^4_1 \qquad \frac{\parallel \quad p}{\neg p \quad \parallel}\neg^4_2 \qquad \frac{\neg p \quad \parallel}{\parallel \quad p}\neg^4_3 \qquad \frac{p \quad \parallel}{\parallel \quad \neg p}\neg^4_4$$

For example, the intuitive reading of rule schema $\vee^4_1$ is that if a formula φ is accepted, then the formula $\varphi \vee \psi$ can also be accepted. The rules containing $\vee$ and $\wedge$ do not involve more than one dimension, however, the case of $\neg$ shows how rejection is internalized using this connective. For instance, $\neg^4_4$ tells us that if we accept φ, we may reject $\neg\varphi$. In a sense that we are about to explain, this system can be shown to induce the same B*-consequence relation as the one induced by the* Σ*-nd-*B*-matrix described in Example 1.*

Clearly, Set^2-Set^2 rule instances are expansors over the node labels algebra $\mathcal{L}_{\mathsf{2D}}$ (check Example 25). Thus, analogously to what we did in the Set-Set case, we take advantage of the contents of Section 3.1 to define derivations in a Set^2-Set^2 system:

Definition 57. *Let $\mathfrak{R}$ be a Set^2-Set^2 system. An $\mathfrak{R}$-*derivation $\mathfrak{t}$ *is a bounded labelled rooted tree in* $\mathsf{Der}(\mathcal{L}_{\mathsf{2D}}, \mathsf{Inst}(\mathfrak{R}))$ *(check Definition 32). Figure 4.1 shows how we graphically represent finite $\mathfrak{R}$-derivations.*

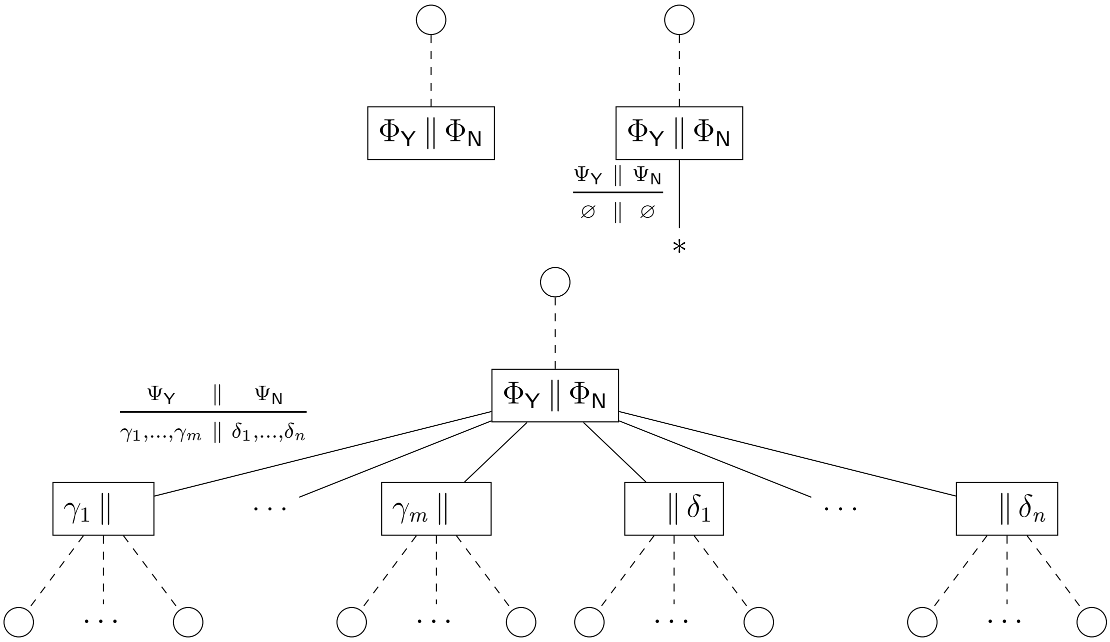


Figure 4.1.: Graphical representation of finite $\mathfrak{R}$–derivations. The dashed edges and blank squares represent other nodes and edges that may exist in the derivation. Notice that we omit the formulas inherited from the parent node, exhibiting only the ones introduced by the applied rule of inference. Also, we have written the rule instance of the rule being applied in the second and third trees, but in practice only the name of the rule of inference will be written for simplicity. In these cases, we emphasize that a precondition for the application of the rule instance is that $\Psi_{\mathsf{Y}} \subseteq \Phi_{\mathsf{Y}}$ and $\Psi_{\mathsf{N}} \subseteq \Phi_{\mathsf{N}}$.

Definition 58. *Let* $\mathfrak{t}$ *be an* $\mathfrak{R}$*-derivation. A node* $\mathfrak{n}$ *of* $\mathfrak{t}$ *is* $(\Psi_{⅄}, \Psi_{И})$-closed *in case it is discontinued (namely, labelled with* $*$*) or it is a leaf node with* $\ell^{\mathfrak{t}}(\mathfrak{n}) = (\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}})$ *and either* $\Phi_{\mathsf{Y}} \cap \Psi_{⅄} \neq \varnothing$ *or* $\Phi_{\mathsf{N}} \cap \Psi_{И} \neq \varnothing$*. A branch of* $\mathfrak{t}$ *is* $(\Psi_{⅄}, \Psi_{И})$-closed *when it ends in a* $(\Psi_{⅄}, \Psi_{И})$*-closed node.*

Definition 59. *An* $\mathfrak{R}$*-derivation* $\mathfrak{t}$ *is* $(\Psi_{⅄}, \Psi_{И})$-closed *when all of its branches are* $(\Psi_{⅄}, \Psi_{И})$*-closed.*

Note that, in the above definitions, we have used the definition of `match` for $\mathcal{L}_{\mathsf{2D}}$, as presented in Example 25. The next definition establishes the meaning of an $\mathfrak{R}$-proof using $\mathfrak{R}$-derivations, being a particular instance of Definition 33.

Definition 60. *An* $\mathfrak{R}$-proof *of* $\left(\begin{array}{c:c} \Phi_{И} & \Phi_{⅄} \\ \hdashline \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}} \end{array}\right)$ *is a* $(\Phi_{⅄}, \Phi_{И})$*-closed* $\mathfrak{R}$*-derivation* $\mathfrak{t}$ *with* $\ell^{\mathfrak{t}}(\mathsf{rt}(\mathfrak{t})) \sqsubseteq$

$(\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}})$.

Example 61. *Below is a proof of* $\left(\begin{array}{c|c} & \neg p \lor \neg q \\ \hdashline \neg(p \land q) & \end{array}\right)$ *in the system* $\mathfrak{R}^{I}$ *introduced in Example 56:*

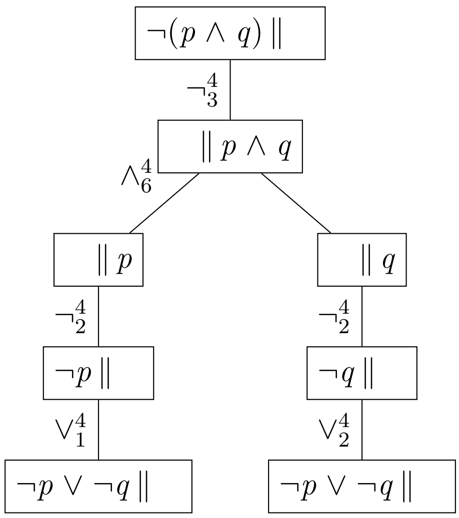


Figure 4.2.: Example of a derivation in tree form. For the sake of a cleaner presentation, we omit the formulas that are inherited when expanding a node.

Given a $\textsc{Set}^2$-$\textsc{Set}^2$ calculus $\mathfrak{R}$, we define the 2×2-place relation $\frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,\mathfrak{R}$ over $\mathsf{Pow}(L_\Sigma(P))$ such that $\frac{\Phi_{\reflectbox{$\mathsf{N}$}}}{\Phi_{\mathsf{Y}}}\big|\frac{\Phi_{\curlywedge}}{\Phi_{\mathsf{N}}}\,\mathfrak{R}$ if, and only if, there is a proof of $\left(\begin{array}{c|c} \Phi_{\reflectbox{N}} & \Phi_{\curlywedge} \\ \hdashline \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}} \end{array}\right)$ in $\mathfrak{R}$. Where R is a $\textsc{Set}^2$-$\textsc{Set}^2$ rule of inference, we may sometimes write $\frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,R$ to refer to the 2×2-place relation induced in the way described above by the $\textsc{Set}^2$-$\textsc{Set}^2$ system containing R as the only rule of inference. This will be of particular interest in the next subsection to facilitate the description of the proof-search algorithm, as well as the proofs of the results related to it. Notice that $\frac{\Phi_{\reflectbox{$\mathsf{N}$}}}{\Phi_{\mathsf{Y}}}\big|\frac{\Phi_{\curlywedge}}{\Phi_{\mathsf{N}}}\,\mathfrak{R}$ if, and only if, $(\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}}) \vdash_{\mathsf{Inst}(\mathfrak{R})} (\Phi_{\curlywedge}, \Phi_{\reflectbox{N}})$, where $\vdash_{\mathsf{Inst}(\mathfrak{R})}$ is as defined in Definition 33. This will be of particular interest for the next proof, most of which is covered by the proof of Proposition 43.

Proposition 62. *The* 2×2*-place relation* $\frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,\mathfrak{R}$ *is the smallest* B*-consequence relation containing the rules of inference of* $\mathfrak{R}$*. If* $\mathfrak{R}$ *is schematic, then* $\frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,\mathfrak{R}$ *is substitution-invariant, and, if it is finitary, then* $\frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,\mathfrak{R}$ *is finitary.*

Proof. In case $\mathfrak{R}$ is schematic, the relation $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}\mathfrak{R}$ is substitution-invariant because $\mathsf{Inst}(\mathfrak{R})$ is closed under substitution instances. Then, it follows directly from Proposition 43 that $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}\mathfrak{R}$ is the least substitution-invariant B-consequence relation containing the rule instances of $\mathfrak{R}$. Suppose now that $\mathfrak{R}$ is finitary and assume that $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}}|\frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\mathfrak{R}$, witnessed by an $\mathfrak{R}$-proof $\mathfrak{t}$, assumed here to be concise. Then (a): $\ell^{\mathfrak{t}}(\mathsf{rt}(\mathfrak{t})) = (\Psi_{\mathsf{Y}}, \Psi_{\mathsf{N}}) \sqsubseteq (\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}})$. In case $\mathfrak{t}$ has a single node, we will have either $\varphi \in \Psi_{\mathsf{Y}} \cap \Phi_{\text{⅄}}$ or $\varphi \in \Psi_{\mathsf{N}} \cap \Phi_{\text{И}}$. In each case, respectively, a labelled rooted tree with a single label labelled with φ witnesses $\frac{}{\varphi}|\frac{\varphi}{}\mathfrak{R}$ or $\frac{\varphi}{}|\frac{}{\varphi}\mathfrak{R}$, as desired. In case $\mathfrak{t}$ has more than one node, we should notice that, since each rule instance of $\mathfrak{R}$ has finitely many formulas in the succedent, $\mathfrak{t}$ has finitely many nodes in each level, thus it has finitely many leaf nodes. Since $\mathfrak{t}$ is concise, each leaf node $\mathfrak{n}$ not labelled with $*$ has a formula that was introduced by the application of the rule instance that produced that very node, which we denote by $\psi_{\mathfrak{n}}$. Without loss of generality, we assume that this formula is in $\Phi_{\text{⅄}} \cup \Phi_{\text{И}}$, since $\mathfrak{t}$ bears witness to $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}}|\frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\mathfrak{R}$. Consider then the sets $\Gamma_{\text{⅄}} := \{\psi_{\mathfrak{n}} \mid \psi_{\mathfrak{n}} \in \Phi_{\text{⅄}}\}$ and $\Gamma_{\text{И}} := \{\psi_{\mathfrak{n}} \mid \psi_{\mathfrak{n}} \in \Phi_{\text{И}}\}$, which are finite subsets of $\Phi_{\text{⅄}}$ and $\Phi_{\text{И}}$, respectively. In addition, let $(\Delta_{\mathsf{Y}}, \Delta_{\mathsf{N}})$ and $(\Delta_{\text{⅄}}, \Delta_{\text{И}})$ be respectively the union of the antecedents and the succedents of all rule instances applied in $\mathfrak{t}$, whose component sets are finite since $\mathfrak{R}$ is finitary. Consider the tree $\mathfrak{t}'$ resulting from $\mathfrak{t}$ by changing, for each node $\mathfrak{n}$, the label of $\mathfrak{n}$ to $\ell^{\mathfrak{t}}(\mathfrak{n}) \cap (\Delta_{\mathsf{Y}} \cup \Delta_{\text{⅄}}, \Delta_{\mathsf{N}} \cup \Delta_{\text{И}})$. Notice that t' is an $\mathfrak{R}$-derivation, and $\ell^{\mathfrak{t}'}(\mathsf{rt}(\mathfrak{t}')) = (\Gamma_{\mathsf{Y}}, \Gamma_{\mathsf{N}})$ has finite component sets. Moreover, Γ_α is finite and $\Gamma_\alpha \subseteq \Phi_\alpha$ for all $\alpha \in \{\mathsf{Y}, \mathsf{N}, \text{⅄}, \text{И}\}$ and $\mathfrak{t}'$ bears witness to $\frac{\Gamma_{\text{И}}}{\Gamma_{\mathsf{Y}}}|\frac{\Gamma_{\text{⅄}}}{\Gamma_{\mathsf{N}}}\mathfrak{R}$, as desired. $\square$

In the proof of the latter result, we have shown how to convert an $\mathfrak{R}$-proof of a B-statement whose nodes might be labelled with infinite sets of formulas into an $\mathfrak{R}$-proof that proves the same statement but having finite sets formulas labelling each node. It turns out that we may convert this tree into a tree with finitely many nodes. In this way, when $\mathfrak{R}$ is finitary, we only need to consider finite $\mathfrak{R}$-derivations:

Corollary 63. *When $\mathfrak{R}$ is finitary, we have $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}}|\frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\mathfrak{R}$ if, and only if, there is a finite tree* *bearing witness to this fact.*

Proof. Suppose that $\mathfrak{t}$ is an $\mathfrak{R}$-proof of $\mathfrak{s} := \left(\begin{array}{c|c} \Phi_{\text{И}} & \Phi_{\text{⅄}} \\ \hline \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}} \end{array}\right)$. Apply the construction described in the proof of Proposition 62 in order to build an $\mathfrak{R}$-proof $\mathfrak{t}'$ of $\mathfrak{s}$ whose nodes are all labelled with finite sets of formulas. If a node in $\mathfrak{t}'$ has infinitely many ancestors, then, since its label is finite, only finitely many of those ancestors were produced by relevant applications of rule instances. We may then apply the procedure described in the proof of Proposition 35 to remove the irrelevant applications of rule instances that lead up to such node, in such a way that its set of ancestors becomes finite. In this way, we may convert $\mathfrak{t}'$ to a finite $\mathfrak{R}$-proof of the same statement. $\square$

Given a $\textsc{Set}^2$-$\textsc{Set}^2$ system $\mathfrak{R}$ and a Σ-nd-B-matrix $\mathfrak{M}$, we say that $\mathfrak{R}$ is *sound* for $\mathfrak{M}$ when $\vdots|\vdots_{\mathfrak{R}} \subseteq \vdots|\vdots_{\mathfrak{M}}$ and that it is *complete* for $\mathfrak{M}$ when the converse holds. When it is both sound and complete for $\mathfrak{M}$, we say that it is *adequate* for $\mathfrak{M}$ or an *axiomatization* for $\mathfrak{M}$.

As we have mentioned, the calculus presented in Example 56 is an axiomatization for the Σ^{FDE}-nd-B-matrix $\mathfrak{M}^{I}$ presented in Example 17. We take now the chance to exemplify the modularity of non-deterministic semantics by illustrating how adding rules to an axiomatization of a nd-B-matrix imposes refinements (that is, restrictions on outputs of some interpretations) in order to guarantee soundness of these very rules. Such mechanism is essential to the axiomatization procedure presented in Chapter 5.

Example 64. *We obtain an axiomatization for $\mathfrak{M}^{E}$ by adding the following rule schemas to the calculus of Example 56:*

$$\frac{p \vee q \quad \|}{p, q \quad \|}\vee_6^4 \qquad \frac{\| \quad p \wedge q}{\| \quad p, q}\wedge_6^4$$

If, in addition, we include the rule schema

$$\frac{q \;\|\; q}{p \;\|\; p}\mathrm{T}^4$$

we axiomatize $\mathfrak{M}^K$ *(see Example 19). Let us explain the intuition behind this mechanism considering the case of schema* $\wedge_6^4$*; the other rules will follow the same principle. What the rule of inference induced by* $\wedge_6^4$ *enforces is that any refinement of* $\mathfrak{M}^I$ *with respect to which this rule is sound must disallow valuations that assign values in* $\{\bot, \mathbf{t}\}$ *to formulas* φ *and* ψ *while assigning a value in* $\{\top, \mathbf{f}\}$ *to* $\varphi \wedge \psi$*, for otherwise such valuation would constitute a countermodel for the instances of that very rule. This is reflected in* $\wedge_{\mathbf{E}}$ *(Example 2) by the absence of the values from the set* $\{\top, \mathbf{f}\}$ *in the entries corresponding to the truth-value assignments in which both inputs belong to* $\{\bot, \mathbf{t}\}$*.*

Example 65. *By the same mechanism used in the previous example, in adding the rules* $\frac{\|}{p \,\|\, p}\bot\mathrm{E}$ *and* $\frac{p \,\|\, p}{\|}\top\mathrm{E}$ *to the axiomatization of* $\mathfrak{M}^E$*, we force empty outputs on any truth-table entry whose input involves either* $\bot$ *or* $\top$*. It follows that Classical Logic inhabits the* t*-aspect of the resulting* Σ*-nd-*B*-matrix, hereby called* $\mathfrak{M}^C$*.*

We emphasize that a Set^2-Set^2 system $\mathfrak{R}$ such that $\vdots|\vdots_{\mathfrak{R}}$ coincides with a given B-consequence relation $\mathsf{C} := \vdots|\vdots$ — we say, in this case, that $\mathfrak{R}$ *axiomatizes* $\vdots|\vdots$ — can be seen as an axiomatization for each of the Set-Set logics associated to $\vdots|\vdots$, since a Set-Set statement can be translated to a B-statement according to the Set-Set logic of interest. For example, if we want to check that $\Phi_{\mathsf{Y}} \rhd_{\mathsf{t}}^{\mathsf{C}} \Phi_{\mathsf{\lambda}}$, we may just check if $\left(\begin{array}{c|c} \varnothing & \Phi_{\mathsf{\lambda}} \\ \hline \Phi_{\mathsf{Y}} & \varnothing \end{array}\right)$ is provable in $\mathfrak{R}$. In Chapter 6, we used this mechanism for providing a finite analytic axiomatization for the Set-Set logic **mCi**. Finally, notice that, in this sense, Set^2-Set^2 systems may axiomatize p-consequences and q-consequences in a natural way, without relying on signals (recall Section 2.5.3).

4.2. Analyticity

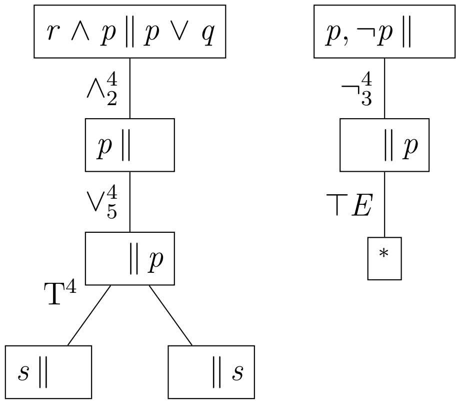
A $\textsc{Set}^2$-$\textsc{Set}^2$ system may give us the guarantee that the formulas appearing in a statement to be proved in that system somehow provides enough material to produce the desired proof, if it exists. The next definitions develop the notion of Θ-*analyticity* for $\textsc{Set}^2$-$\textsc{Set}^2$ systems, which is a way of expressing such a guarantee. They are essentially adaptations of this same notion for $\textsc{Set}$-$\textsc{Set}$ systems, established and studied recently in [17, 43].

We denote by $\mathsf{fmlas}(\mathfrak{t})$ the set of formulas occurring as node labels in an $\mathfrak{R}$-tree $\mathfrak{t}$, that is, $\mathsf{fmlas}(\mathfrak{t}) := \bigcup_{\mathfrak{n} \in \mathsf{nds}(\mathfrak{t})} \ell^{\mathfrak{t}}(\mathfrak{n})$. Given a B-statement $\mathfrak{s} := \left(\begin{array}{c:c} \Phi_{\text{И}} & \Phi_{\lambda} \\ \hdashline \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}} \end{array}\right)$, we let $\mathsf{subf}(\mathfrak{s}) := \bigcup_{\alpha} \mathsf{subf}[\Phi_\alpha]$ be the collection of *subformulas of* $\mathfrak{s}$. In addition, provided that Θ is a set of unary formulas (that is, formulas on a single propositional variable), we let $\mathsf{gsubf}^{\Theta}(\mathfrak{s}) := \mathsf{subf}(\mathfrak{s}) \cup \{\sigma(\varphi) \mid \varphi \in \Theta, \sigma : P \to \mathsf{subf}(\mathfrak{s})\}$, the collection of Θ-*instantiated subformulas* of $\mathfrak{s}$, again a notion adapted from [17, 43].

Example 66. *Consider the set $\Theta := \{p, \neg p\}$ of unary formulas over a signature containing a unary connective $\neg$ and a binary connective $\wedge$. Then the set of Θ-instantiated subformulas of the* B*-statement* $\left(\begin{array}{c:c} r & \\ \hdashline q \wedge \neg s & r \end{array}\right)$ *is* $\{q, r, s, \neg s, q \wedge \neg s, \neg q, \neg r, \neg\neg s, \neg(q \wedge \neg s)\}$.

Definition 67. *An $\mathfrak{R}$-proof $\mathfrak{t}$ of* $\mathfrak{s} := \left(\begin{array}{c:c} \Phi_{\text{И}} & \Phi_{\lambda} \\ \hdashline \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}} \end{array}\right)$ *is* Θ-analytic *when* $\mathsf{fmlas}(\mathfrak{t}) \subseteq \mathsf{gsubf}^{\Theta}(\mathfrak{s})$.

Example 68. *We provide here $\{p\}$-analytic proofs of the statements* $\left(\begin{array}{c:c} s & s \\ \hdashline r \wedge p & p \vee q \end{array}\right)$ *and* $\left(\begin{array}{c:c} & \\ \hdashline p, \neg p & \end{array}\right)$*, respectively, in the calculi for $\mathfrak{M}^K$ and $\mathfrak{M}^C$ presented in the previous examples. One should also note that the proof given in Example 61 is also $\{p\}$-analytic.*



When there is a Θ-analytic $\mathfrak{R}$-proof of $\left(\frac{\Phi_{\text{ꟽ}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\right)$, we write $\frac{\Phi_{\text{ꟽ}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\,{}^{\Theta}_{\mathfrak{R}}$. Similar to Proposition 62, we have:

Proposition 69. *The* 2×2*-place relation* $\frac{\cdot}{\cdot}\Big|\frac{\cdot}{\cdot}\,{}^{\Theta}_{\mathfrak{R}}$ *is a* B*-consequence relation.*

Definition 70. *A* Set^2-Set^2 *calculus* $\mathfrak{R}$ *is* Θ-analytic *when* $\frac{\cdot}{\cdot}\Big|\frac{\cdot}{\cdot}\,\mathfrak{R} \subseteq \frac{\cdot}{\cdot}\Big|\frac{\cdot}{\cdot}\,{}^{\Theta}_{\mathfrak{R}}$ *(notice that the converse always hold).*

In Chapter 5, we will see that Θ-analytic Set^2-Set^2 systems are not rare, inconceivable creatures; they are actually pretty common: there is one for each Σ-nd-B-matrix satisfying a (very inclusive) sufficient expressiveness requirement. Before going into that, we will introduce and study a straightforward exponential proof-search procedure for finite and finitary Set^2-Set^2 systems.

4.3. A proof-search and countermodel-search algorithm

Let $\mathfrak{s} := \left(\frac{\Phi_{\text{ꟽ}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\right)$ be an arbitrary Set^2-Set^2 statement, $\mathfrak{R}$ be a finite and finitary Set^2-Set^2 system, and Θ be a finite set of unary formulas. Notice that, whenever $\mathfrak{R}$ is Θ-analytic, it is enough to consider the rule instances in $\mathfrak{R}[\mathfrak{s}]$ in order to provide a proof of $\mathfrak{s}$ in $\mathfrak{R}$. Searching for such a proof is clearly a particular case of finding a proof of $\mathfrak{s}$ using only candidates in a finite set R of finitary rule instances. A proof-search algorithm for this more general setting is readily available and its pseudocode is presented in Algorithm 4.1 by means of a function called Proof-Search. It is essentially a version for the Set^2-Set^2 environment of the procedure described in the concluding remarks of [43]. The algorithm searches for a proof by expanding nodes that are not closed or discontinued using only instances in R that were not used yet in the branch of the node under expansion. As we shall see in the sequel, the order in which applicable instances are selected does not affect the result, although for sure smarter choice heuristics may well improve the performance of the algorithm in particular cases.

Algorithm 4.1: Proof search over a finite set of finitary rule instances

function Proof-Search($F := (\Psi_{\mathsf{Y}}, \Psi_{\mathsf{N}}),\ C := (\Phi_{\text{⅄}}, \Phi_{\text{И}}),\ R$)**:**
    Input: antecedent in F, succedent in C and a finite set R of finitary rule instances
    $\mathfrak{t} \leftarrow$ a tree with a single node labelled with $(\Psi_{\mathsf{Y}}, \Psi_{\mathsf{N}})$
    if $\Psi_\alpha \cap \Phi_{\tilde{\alpha}} \neq \varnothing$ *for some* $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$ **then return** $\mathfrak{t}$
    foreach *rule instance* $r := \frac{\Delta_{\mathsf{Y}} \,\|\, \Delta_{\mathsf{N}}}{\Delta_{\text{⅄}} \,\|\, \Delta_{\text{И}}} \in R$ **do**
        if $\Delta_{\tilde{\alpha}} \cap \Psi_\alpha = \varnothing$ *and* $\Delta_\alpha \subseteq \Psi_\alpha$ *for each* $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$ **then**
            if $\Delta_{\text{⅄}} \cup \Delta_{\text{И}} = \varnothing$ **then return** *the tree resulting from adding a node labelled with* $*$ *as the single child of* $\mathsf{rt}(\mathfrak{t})$.
            foreach $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$ *and* $\varphi \in \Delta_{\tilde{\alpha}}$ **do**
                $\mathfrak{t}' \leftarrow$ Proof-Search($(\Psi_{\mathsf{Y}} \cup P_{\mathsf{Y}}(\varphi), \Psi_{\mathsf{N}} \cup P_{\mathsf{N}}(\varphi)),\ C,\ R \backslash \{r\}$), where $P_\alpha(\varphi)$ is $\varnothing$ if $\varphi \notin \Delta_{\tilde{\alpha}}$ and $\{\varphi\}$ otherwise
                $\mathfrak{t} \leftarrow$ the tree resulting from adding $\mathfrak{t}'$ as a subtree of $\mathfrak{t}$, with the root of the latter being the parent of the root of the former.
                if $\mathfrak{t}'$ *is not* C*-closed* **then return** $\mathfrak{t}$
            end
            if $\mathfrak{t}$ *is* C*-closed* **then return** $\mathfrak{t}$
        end
    end
    return $\mathfrak{t}$

When the proof-search algorithm produces a tree that is not C-closed, we say that it is C-*open*, or, when there is no risk of confusion, just *open*. The following lemma proves the termination of Proof-Search and its correctness. The subsequent result establishes the applicability of this algorithm for proof search over Θ-analytic calculi.

Lemma 71. *Let $\mathfrak{R}$ be a finite and finitary* Set^2-Set^2 *system, whose rules of inference are all finite, that is,* $\mathsf{Inst}(\mathfrak{R})$ *is a finite set of finitary rule instances. Then the procedure* Proof-Search$((\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}}), (\Phi_{\mathsf{⅄}}, \Phi_{\mathsf{И}}), \mathsf{Inst}(\mathfrak{R}))$ *always terminates, returning an $\mathfrak{R}$-tree that is $(\Phi_{\mathsf{⅄}}, \Phi_{\mathsf{И}})$-closed if, and only if, $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}} | \frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}} \mathfrak{R}$.*

Proof. Let $\mathfrak{R}$ be a system as specified in the statement, set $F := (\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}})$ and set $C := (\Phi_{\mathsf{⅄}}, \Phi_{\mathsf{И}})$. Also, let $R := \mathsf{Inst}(\mathfrak{R})$. We proceed by induction on $|R|$. In the base case, $R = \varnothing$, the algorithm obviously terminates and returns a proof of $\mathfrak{s}$ iff $\frac{\Phi_{\mathsf{Y}} \parallel \Phi_{\mathsf{N}}}{\Phi_{\mathsf{⅄}} \parallel \Phi_{\mathsf{И}}} \varnothing$. In the inductive step, assume that $|R| \geq 1$ and that (IH): the present lemma holds for all sets of rule instances R' with $|R'| = |R| - 1$. Since R is finite and contains only finitary rule instances, and each recursive call (line 8) terminates by (IH), the whole algorithm terminates. Also, if a C-closed tree is produced, it means that one of the conditions in lines 3, 6 or 12 was satisfied. The first possibility (line 3) was treated in the base case. The second one (line 6) means that there is a rule instance in R with an empty succedent satisfying the antecedents, in which case a tree with its root labelled with F having a single child labelled with $*$ is returned, clearly bearing witness to $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}} | \frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}} R$. The third possibility (line 12) means that there is a rule instance $r := \frac{\Delta_{\mathsf{Y}} \parallel \Delta_{\mathsf{N}}}{\Delta_{\mathsf{⅄}} \parallel \Delta_{\mathsf{И}}} \in R$ applicable to the antecedents in F (line 5), and, by (IH), the recursive calls (line 8) produce trees that bear witness to $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}},\varphi} | \frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}} R \backslash \{r\}$ and $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}} | \frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}},\psi} R \backslash \{r\}$ for each $\varphi \in \Delta_{\mathsf{⅄}}$ and $\psi \in \Delta_{\mathsf{И}}$. The resulting tree, then, bears witness to $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}} | \frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}} R$. On the other hand, if an open tree is produced, then for a rule instance $r := \frac{\Delta_{\mathsf{Y}} \parallel \Delta_{\mathsf{N}}}{\Delta_{\mathsf{⅄}} \parallel \Delta_{\mathsf{И}}} \in R$ applicable to F, some recursive call resulted in an open tree. Assume, without loss of generality, that such call referred to an expansion by $\varphi \in \Delta_{\mathsf{⅄}}$. Then, by (IH), $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}},\varphi} \nmid \frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}} R \backslash \{r\}$. Because $\varphi \in \Delta_{\mathsf{⅄}}$, the instance

r does not play any role in deriving $\left(\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y},\varphi}} \,\Big|\, \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\right)$, so we have $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y},\varphi}} \nmid \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\, R$ and, by **(D2)**, it follows that $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \nmid \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\, R$. ☐

A B-consequence $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}$ is said to be *decidable* when there is some *decision procedure* that takes a B-statement $\left(\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \,\Big|\, \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\right)$ as input and outputs `true` when $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} | \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}$ is the case, and outputs `false` when $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \nmid \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}$.

Lemma 72. *If $\mathfrak{R}$ is a finite and finitary Θ-analytic* SET^2-SET^2 *system, then* PROOF-SEARCH *is a proof-search algorithm for $\mathfrak{R}$ and a decision procedure for $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}\,\mathfrak{R}$.*

Proof. We know that $\mathfrak{R}[\mathfrak{s}]$ must be enough to provide a derivation of $\mathfrak{s}$, since $\mathfrak{R}$ is Θ-analytic. Clearly, such set is finite and contains only finitary rule instances, hence the present result is a direct consequence of Lemma 71. ☐

In what follows, let $\mathfrak{s} := \left(\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \,\Big|\, \frac{\Phi_{\text{И}}}{\Phi_{\mathsf{N}}}\right)$ be a B-statement and $\mathsf{size}(\mathfrak{s}) := \sum_{\alpha} \mathsf{size}[\Phi_{\alpha}]$ be the *size* of $\mathfrak{s}$.

Lemma 73. *Let $\mathfrak{R}$ be a finite* SET^2-SET^2 *calculus, with finite and finitary rules of inference. Then the worst-case running time of* PROOF-SEARCH$((\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}}), (\Phi_{\text{⅄}}, \Phi_{\text{И}}), \mathsf{Inst}(\mathfrak{R}))$ *is $O(b^n + n \cdot \mathsf{p}(s))$, where $b := \max_{r \in \mathsf{Inst}(\mathfrak{R})} \mathsf{branch}(r)$, $s := \mathsf{size}(\mathfrak{s})$ and $n := |\mathsf{Inst}(\mathfrak{R})|$.*

Proof. The worst-case running-time $T(n, s)$ of PROOF-SEARCH occurs when $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} | \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\, R$, the set R needs to be entirely inspected until an applicable rule instance is found, and such an instance does not have an empty set of succedents. The following table assigns a cost and simplified upper bounds to execution times of the relevant instructions of Algorithm 4.1 in the described scenario.

Instruction line	Cost	Times
2	c_1	1
3	$\mathsf{p}(s)$	1
4	c_2	n
5	$\mathsf{p}(s)$	n
6	c_3	1
7	c_4	b
8	$T(n-1, s+\mathsf{p}(s))$	b
9	c_5	b
10	$\mathsf{p}(s)$	b
12	$\mathsf{p}(s)$	1

Notice that $T(0, s) = c_1 + \mathsf{p}(s)$ and, based on the assignments above and after some algebraic manipulations, we have, for $n \geq 1$,

$$T(n, s) \leq b \cdot T(n-1, s+\mathsf{p}(s)) + 2n \cdot \mathsf{p}(s). \tag{4.1}$$

We prove by induction on n that $T(n, s) \in O(b^n + n \cdot \mathsf{p}(s))$. We will take advantage of the asymptotic notation and choose the base case as $n = 1$. In such case, we have $T(1, s) \leq bT(0, s+\mathsf{p}(s)) + 2\mathsf{p}(s) = 2\mathsf{p}(s) + bc_1 + b\mathsf{p}(s) = c_1 b + (2+b)\mathsf{p}(s)$, and the upper bound suffices. In the inductive step, let $n > 1$ and assume, for all $s \geq 0$, that

$T(n-1, s) \leq k_1 \cdot b^{n-1} + k_1 \cdot (n-1) \cdot \mathsf{p}(s)$, for some $k_1 > 0$. Then

$$\begin{aligned}
T(n,s) &\leq b \cdot T(n-1, s + \mathsf{p}(s)) + 2n \cdot \mathsf{p}(s) \\
&\leq b \cdot (k_1 \cdot b^{n-1} + k_1 \cdot (n-1) \cdot \mathsf{p}(s)) + 2n \cdot \mathsf{p}(s) \\
&= k_1 \cdot b^n + b \cdot k_1 \cdot (n-1) \cdot \mathsf{p}(s) + 2n \cdot \mathsf{p}(s) \\
&= k_1 \cdot b^n + (2n + b \cdot k_1 \cdot n - b \cdot k_1) \cdot \mathsf{p}(s) \\
&\leq k_1 \cdot b^n + (2n + b \cdot k_1 \cdot n) \cdot \mathsf{p}(s) \\
&= k_1 \cdot b^n + k_2 \cdot n \cdot \mathsf{p}(s), \text{ with } k_2 = 2 + b \cdot k_1 \\
&\leq k_3 \cdot (b^n + n \cdot \mathsf{p}(s)), \text{ with } k_3 = \max\{k_1, k_2\} \\
&\in O(b^n + n \cdot \mathsf{p}(s))
\end{aligned}$$

□

Theorem 74. *If $\mathfrak{R}$ is a finite and Θ-analytic* Set^2-Set^2 *system with finite and finitary rules of inference,* Proof-Search *is a proof-search algorithm for $\mathfrak{R}$ that runs in exponential time in general, and in polynomial time if $\mathfrak{R}$ contains only rules with at most one formula in the succedent.*

Proof. Clearly, the set of all instances of rules of $\mathfrak{R}$ using only formulas in $\mathsf{gsubf}^{\Theta}(\mathfrak{s})$ is finite and contains only finitary rule instances, and its size is polynomial in $\mathsf{size}(\mathfrak{s})$. The announced result then follows directly from Lemma 73. □

A branch of an C-open tree $\mathfrak{t}$ is itself *open* when it ends in a node which is not C-closed. Open branches in a tree resulting from a failed attempt of finding a proof of a Set^2-Set^2 statement $\mathfrak{s}$ using Algorithm 4.1 may help us in coming up with a semantical countermodel for $\mathfrak{s}$, provided we have enough ingredients in hand. As we explain in the proof of the proposition below, the label of the leaf node of an open branch of the said tree gives us partitions on the set of Θ-instantiated subformulas of $\mathfrak{s}$. In the next chapter, we will produce SET^2-SET^2 systems having a semantical counterpart in terms of a single nd-B-matrix in which there is a way to assign values to each subformula of the formulas in $\mathfrak{s}$ based on the obtained partitions. In this way, we will be able to provide a valuation on that nd-B-matrix witnessing the non-provability of the statement.

Proposition 75. *Suppose that $\mathfrak{R}$ is Θ-analytic. Let $\mathfrak{t}$ result from a failed proof search attempt of the* B*-statement* $\mathfrak{s} := \left(\begin{smallmatrix}\Phi_{\text{И}} & \Phi_{\text{⅄}} \\ \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}}\end{smallmatrix}\right)$ *using Algorithm 4.1. Then, from $\mathfrak{t}$, we may extract sets $\Psi_{\mathsf{S}}, \Psi_{\text{Ƨ}} \subseteq \mathsf{gsubf}^{\Theta}(\mathfrak{s})$ such that* $\frac{\mathsf{gsubf}^{\Theta}(\mathfrak{s})\setminus\Psi_{\text{Ƨ}}}{\Psi_{\mathsf{S}}} \nmid \frac{\mathsf{gsubf}^{\Theta}(\mathfrak{s})\setminus\Psi_{\mathsf{S}}}{\Psi_{\text{Ƨ}}}\ \mathfrak{R}$.

Proof. Since the proof search has failed, the produced tree $\mathfrak{t}$ must have an open branch b. Let $(\Psi_{\mathsf{S}}, \Psi_{\text{Ƨ}})$ be the label of the leaf node of b. As the algorithm searches for a Θ-analytic proof, we have $\Psi_{\mathsf{S}}, \Psi_{\text{Ƨ}} \subseteq \mathsf{gsubf}^{\Theta}(\mathfrak{s})$. Moreover, the fact that the branch is open means that it is not amenable to a relevant expansion by any of the rule instances of $\mathfrak{R}$ whose formulas are in $\mathsf{gsubf}^{\Theta}(\mathfrak{s})$. In this way, we have $\frac{\mathsf{gsubf}^{\Theta}(\mathfrak{s})\setminus\Psi_{\text{Ƨ}}}{\Psi_{\mathsf{S}}} \nmid \frac{\mathsf{gsubf}^{\Theta}(\mathfrak{s})\setminus\Psi_{\mathsf{S}}}{\Psi_{\text{Ƨ}}}\ \overset{\Theta}{\mathfrak{R}}$, and thus $\frac{\mathsf{gsubf}^{\Theta}(\mathfrak{s})\setminus\Psi_{\text{Ƨ}}}{\Psi_{\mathsf{S}}} \nmid \frac{\mathsf{gsubf}^{\Theta}(\mathfrak{s})\setminus\Psi_{\mathsf{S}}}{\Psi_{\text{Ƨ}}}\ \mathfrak{R}$, since $\mathfrak{R}$ is Θ-analytic. $\square$

It turns out that the pair of sets mentioned in the above result may be guessed in time polynomial in the size of the B-statement $\mathfrak{s}$. This allows us to prove the following result:

Theorem 76. *If $\mathfrak{R}$ is Θ-analytic, then the problem of deciding $\div|\div\ \mathfrak{R}$ is in* coNP.

Proof. Let $\mathfrak{s} := \left(\begin{smallmatrix}\Phi_{\text{И}} & \Phi_{\text{⅄}} \\ \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}}\end{smallmatrix}\right)$. Given a pair $(\Psi_{\mathsf{Y}}, \Psi_{\mathsf{N}})$ with $\Psi_{\mathsf{Y}} \cup \Psi_{\mathsf{N}} \subseteq \mathsf{gsubf}^{\Theta}(\mathfrak{s})$, $\Phi_\alpha \subseteq \Psi_\alpha$ and $\Phi_{\tilde{\alpha}} \cap \Psi_\alpha = \varnothing$ for each $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$, if we check that for every applicable rule instance $\frac{\Delta_{\mathsf{Y}} \parallel \Delta_{\mathsf{N}}}{\Delta_{\text{⅄}} \parallel \Delta_{\text{И}}} \in \mathfrak{R}[\mathfrak{s}]$ we have $\Psi_\alpha \cap \Delta_{\tilde{\alpha}} \neq \varnothing$ for each $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$, then $\frac{\mathsf{gsubf}^{\Theta}(\mathfrak{s})\setminus\Psi_{\mathsf{N}}}{\Psi_{\mathsf{Y}}} \nmid \frac{\mathsf{gsubf}^{\Theta}(\mathfrak{s})\setminus\Psi_{\mathsf{Y}}}{\Psi_{\mathsf{N}}}\ \overset{\Theta}{\mathfrak{R}}$, and thus, by **(D2)**, $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \nmid \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\ \overset{\Theta}{\mathfrak{R}}$. Since the amount of rule instances is polynomial in the size of $\mathfrak{s}$, by guessing in polynomial time that pair $(\Psi_{\mathsf{Y}}, \Psi_{\mathsf{N}})$ and performing the described test we obtain a polynomial-time non-deterministic algorithm to verify whether $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \nmid \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\ \overset{\Theta}{\mathfrak{R}}$, and so the problem of deciding $\div|\div\ \mathfrak{R}$ is in coNP. $\square$

5. Analytic H-systems for non-deterministic B-matrices

5.1. Sufficient expressiveness

We are interested here in formulating for Σ-nd-B-matrices a notion analogous to that of *sufficient expressiveness* for Σ-nd-matrices, which is commonly referred to as *monadicity* in the literature [57, 43, 17]. Such property entails that every truth-value can be characterized by way of unary formulas $\mathrm{S}(p)$ in the object-language, in the sense that a valuation assigns a value x to a formula φ whenever it behaves in a specific way with respect to each $\mathrm{S}(\varphi)$. In a moment, we will see that Σ-nd-B-matrices satisfying this property are all axiomatizable in terms of analytic $\textsc{Set}^2$-$\textsc{Set}^2$ systems by means of a procedure that is easy to implement in the case of finite matrices.

Definition 77. *Let $\mathfrak{M} := \langle \mathbf{A}, \mathsf{Y}, \mathsf{N} \rangle$ be a Σ-nd-B-matrix.*

- *Given $X, Y \subseteq A$ and $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$, we say that X and Y are* α-separated*, denoted by $X \#_\alpha Y$, if $X \subseteq \alpha$ and $Y \subseteq \tilde{\alpha}$, or vice-versa.*
- *Given distinct truth-values $x, y \in A$, a unary formula S is a* separator for (x, y) *whenever $\mathrm{S}_{\mathbf{A}}(x) \#_\alpha \mathrm{S}_{\mathbf{A}}(y)$ for some $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$. If for each pair of distinct truth-values in A there is a separator for these values, then $\mathfrak{M}$ is said to be* sufficiently expressive*.*
- *A set of unary formulas $\mathcal{D}^x$* isolates *$x \in A$ whenever, for every $y \neq x$, there exists a*

separator in $\mathcal{D}^x$ for x and y.

- *A* discriminator for $\mathfrak{M}$ *is a family* $\mathcal{D} := \{(\mathcal{D}^x_{\mathsf{Y}}, \mathcal{D}^x_{\mathsf{\lambda}}, \mathcal{D}^x_{\mathsf{N}}, \mathcal{D}^x_{\mathsf{И}})\}_{x \in A}$ *such that* $\mathcal{D}^x := \bigcup_{\alpha \in \{\mathsf{Y},\mathsf{\lambda},\mathsf{N},\mathsf{И}\}} \mathcal{D}^x_\alpha$ *isolates* x *and* $\mathrm{S}_{\mathbf{A}}(x) \subseteq \alpha$ *whenever* $\mathrm{S} \in \mathcal{D}^x_\alpha$*. We denote the set* $\bigcup_{x \in A} \bigcup_{\alpha \in \{\mathsf{Y},\mathsf{\lambda},\mathsf{N},\mathsf{И}\}} \mathcal{D}^x_\alpha$ *by* $\mathcal{D}^{\bowtie}$ *and say that* $\mathcal{D}$ *is* based on $\mathcal{D}^{\bowtie}$.

Example 78. *The tables below describe, respectively, a discriminator based on* $\{p\}$ *for any* Σ*-nd-*B*-matrix of the form* $\langle \mathcal{V}_4, \mathsf{Y}_4, \mathsf{N}_4 \rangle$ *(see Examples 1, 2 and 3), and a discriminator for* $\mathfrak{M}_{\mathbf{mCi}}$ *(recall Example 20) based on* $\{p, \neg p\}$*:*

x	$\mathcal{D}^x_{\mathsf{Y}}$	$\mathcal{D}^x_{\mathsf{\lambda}}$	$\mathcal{D}^x_{\mathsf{N}}$	$\mathcal{D}^x_{\mathsf{И}}$
$\mathbf{f}$	$\varnothing$	p	p	$\varnothing$
$\bot$	$\varnothing$	p	$\varnothing$	p
$\top$	p	$\varnothing$	p	$\varnothing$
$\mathbf{t}$	p	$\varnothing$	$\varnothing$	p

x	$\mathcal{D}^x_{\mathsf{Y}}$	$\mathcal{D}^x_{\mathsf{\lambda}}$	$\mathcal{D}^x_{\mathsf{N}}$	$\mathcal{D}^x_{\mathsf{И}}$
f	$\varnothing$	p	p	$\varnothing$
F	$\varnothing$	p	$\varnothing$	p
I	$p, \neg p$	$\varnothing$	p	$\varnothing$
T	p	$\neg p$	p	$\varnothing$
t	p	$\varnothing$	$\varnothing$	p

Let us look at the rightmost table. Each row corresponds to a truth-value x and describes the sets $\mathcal{D}^x_\alpha$*, for each* $\alpha \in \{\mathsf{Y},\mathsf{\lambda},\mathsf{N},\mathsf{И}\}$*. Consider, for instance, the value I. The table indicates that* $\mathcal{D}^I_{\mathsf{Y}} = \{p, \neg p\}$*,* $\mathcal{D}^I_{\mathsf{\lambda}} = \varnothing$*,* $\mathcal{D}^I_{\mathsf{N}} = \{p\}$ *and* $\mathcal{D}^I_{\mathsf{И}} = \varnothing$*. The union of these sets isolates I, since p is a separator for* (I, f)*,* (I, F) *and* (I, t)*, and* $\neg p$ *is a separator for* (I, T) *in* $\mathfrak{M}_{\mathbf{mCi}}$*.*

We should emphasize here the significance of *not* requiring that each pair of truth-values ought to be separable with respect to *both* distinguished sets of truth-values in order to characterize an nd-B-matrix as sufficiently expressive (note that, in the definition of separator presented above, we have used “for some $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$” instead of “for each $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$”). This stronger alternative would be too restrictive, hiding the power gained by having an additional dimension in the matrix structure. As we will see in Chapter 6, a pair of truth-values might not be separable with respect to one of the dimensions, while being separable with respect to the other one. Despite that, when being able to separate each pair of truth-values with respect to *at least* one of the dimensions, we are still capable of characterizing each truth-value of the nd-B-matrix in hand, and, as we will see in a moment, algorithmically axiomatize it.

The following result — which will be instrumental, in particular, within the soundness proof of the axiomatizations that we will develop later on — shows that a discriminator is capable of uniquely characterizing each truth-value of the corresponding Σ-nd-B-matrix:

Lemma 79. *If $\mathfrak{M} := \langle \mathbf{A}, \mathsf{Y}, \mathsf{N}\rangle$ is a sufficiently expressive Σ-nd-B-matrix and $\mathcal{D}$ is a discriminator for $\mathfrak{M}$, then, for all $\varphi \in L_\Sigma(P)$, $x \in A$ and $\mathfrak{M}$-valuation v,*

$$v(\varphi) = x \;\; \textit{iff} \;\; v[\mathcal{D}^x_\alpha(\varphi)] \subseteq \alpha \textit{ and } v[\mathcal{D}^x_{\tilde{\alpha}}(\varphi)] \subseteq \tilde{\alpha} \textit{ for every } \alpha \in \{\mathsf{Y}, \mathsf{N}\}.$$

Proof. From the left to the right, assume that $v(\varphi) = x$ and let $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$. If $\mathrm{S} \in \mathcal{D}^x_\alpha$, then $v(\mathrm{S}(\varphi)) \in \mathrm{S}_{\mathbf{A}}(v(\varphi)) = \mathrm{S}_{\mathbf{A}}(x)$, and we know that $\mathrm{S}_{\mathbf{A}}(x) \subseteq \alpha$ if $\mathrm{S} \in \mathcal{D}^x_\alpha$. The same reasoning applies for $\mathrm{S} \in \mathcal{D}^x_{\tilde{\alpha}}$. Conversely, we may argue contrapositively: suppose that $v(\varphi) = y \neq x$ and consider the separator $\mathrm{S} \in \mathcal{D}^x$ for x and y, such that $\mathrm{S}_{\mathbf{A}}(x) \#_\alpha \mathrm{S}_{\mathbf{A}}(y)$, for some $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$. By cases, if $\mathrm{S} \in \mathcal{D}^x_\alpha$, then $v(\mathrm{S}(\varphi)) \in \mathrm{S}_{\mathbf{A}}(y) \subseteq \tilde{\alpha}$ and so $v(\mathcal{D}^x_\alpha(\varphi)) \not\subseteq \alpha$; analogously, we have $v(\mathcal{D}^x_{\tilde{\alpha}}(\varphi)) \not\subseteq \tilde{\alpha}$ if $\mathrm{S} \in \mathcal{D}^x_{\tilde{\alpha}}$. $\square$

5.2. Axiomatizing non-deterministic B-matrices

We now describe four collections of rule schemas by which any sufficiently expressive Σ-nd-B-matrix $\mathfrak{M}$ is constrained. Together, these schemas constitute a presentation of a calculus that will be denoted by $\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$, where $\mathcal{D}$ is a discriminator for $\mathfrak{M}$. The first collection, ($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\exists}$), is intended to exclude all combinations of separators that do not correspond to truth-values. The second, ($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathsf{D}}$), sets the combinations of separators that characterize acceptance apart from those that characterize non-acceptance, and sets the combinations of separators that characterize rejection apart from those that characterize non-rejection. The third one, ($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\Sigma}$), fully describes, through appropriate refinements, the interpretation of the connectives of Σ in $\mathfrak{M}$. At last, the rules in ($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathbb{T}}$) guarantee that values belong to total sub-Σ-nd-B-matrices of $\mathfrak{M}$.

In what follows, given $X \subseteq A$, we shall use $(\dot{\mathcal{D}}^X_{\mathsf{Y}}, \dot{\mathcal{D}}^X_{\mathsf{N}})$ to denote a pair of sets in which $\dot{\mathcal{D}}^X_{\alpha}$, with $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$, is obtained by choosing an element of $\mathcal{D}^x_{\alpha}$ for each $x \in X$. When $X = \varnothing$, the only possibility is the pair $(\varnothing, \varnothing)$; moreover, when $\mathcal{D}^x_{\mathsf{Y}} \cup \mathcal{D}^x_{\mathsf{N}} = \varnothing$ for some $x \in X$, no such pair exists. We shall use $(\dot{\mathcal{D}}^X_{\text{⅄}}, \dot{\mathcal{D}}^X_{\text{И}})$ analogously.

Example 80. *Consider the discriminator for $\mathfrak{M}_{\mathbf{mCi}}$ presented in Example 78. For $X := \{I, T\}$, the possible pairs $(\dot{\mathcal{D}}^X_{\mathsf{Y}}, \dot{\mathcal{D}}^X_{\mathsf{N}})$ that we may choose in this situation are $(\{p\}, \{p\})$ and $(\{\neg p, p\}, \{p\})$, whilst there is a single possible pair $(\dot{\mathcal{D}}^X_{\text{⅄}}, \dot{\mathcal{D}}^X_{\text{И}})$, which is $(\{p\}, \varnothing)$. For $X := \{F\}$, there is no choice for $(\dot{\mathcal{D}}^X_{\mathsf{Y}}, \dot{\mathcal{D}}^X_{\mathsf{N}})$.*

We are now ready to introduce the recipe for axiomatizing sufficiently expressive Σ-nd-B-matrices. Notice that we do not constrain the nd-B-matrix to be finite.

Definition 81. *Let $\mathfrak{M} := \langle \mathbf{A}, \mathsf{Y}, \mathsf{N} \rangle$ be a Σ-nd-B-matrix and $\mathcal{D}$ be a discriminator for $\mathfrak{M}$. The* Set^2-Set^2 *calculus $\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$ is presented by way of the following rule schemas:*

($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\exists}$) *for each $X_1 \subseteq A$ and each possible choices of $(\dot{\mathcal{D}}^{X_0}_{\mathsf{Y}}, \dot{\mathcal{D}}^{X_0}_{\mathsf{N}})$ and of $(\dot{\mathcal{D}}^{X_1}_{\text{⅄}}, \dot{\mathcal{D}}^{X_1}_{\text{И}})$, with $X_0 := A \backslash X_1$,*

$$\frac{\dot{\mathcal{D}}^{X_1}_{\text{⅄}} \quad \| \quad \dot{\mathcal{D}}^{X_1}_{\text{И}}}{\dot{\mathcal{D}}^{X_0}_{\mathsf{Y}} \quad \| \quad \dot{\mathcal{D}}^{X_0}_{\mathsf{N}}}$$

($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathsf{D}}$) *for an arbitrary propositional variable $p \in P$, and for each $x \in A$,*

$$\frac{\mathcal{D}^x_{\mathsf{Y}}(p), \mathsf{p}_{\text{⅄}}(x) \quad \| \quad \mathcal{D}^x_{\mathsf{N}}(p)}{\mathcal{D}^x_{\text{⅄}}(p), \mathsf{p}_{\mathsf{Y}}(x) \quad \| \quad \mathcal{D}^x_{\text{И}}(p)} \qquad \frac{\mathcal{D}^x_{\mathsf{Y}}(p) \quad \| \quad \mathcal{D}^x_{\mathsf{N}}(p), \mathsf{p}_{\text{И}}(x)}{\mathcal{D}^x_{\text{⅄}}(p) \quad \| \quad \mathcal{D}^x_{\text{И}}(p), \mathsf{p}_{\mathsf{N}}(x)}$$

where, for $\alpha \in \{\mathsf{Y}, \mathsf{N}, \text{⅄}, \text{И}\}$, $\mathsf{p}_{\alpha} : A \to \mathsf{Pow}(\{p\})$ is such that $\mathsf{p}_{\alpha}(x) = \{p\}$ iff $x \in \alpha$.

($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\Sigma}$) *for each k-ary connective ©, each sequence $X := (x_1, \ldots, x_k)$ of truth-values of*

$\mathfrak{M}$*, each* $y \notin ©_{\mathbf{A}} X$*, and for a sequence* $(p_1, \ldots, p_k)$ *of distinct propositional variables,*

$$\frac{\Theta_{\mathsf{Y}}^{©,X,y} \quad \| \quad \Theta_{\mathsf{N}}^{©,X,y}}{\Theta_{\text{⅄}}^{©,X,y} \quad \| \quad \Theta_{\text{И}}^{©,X,y}} \; ©_y^X$$

where each $\Theta_{\alpha}^{©,x_1,\ldots,x_k,y} := \bigcup_{1 \leq i \leq k} \mathcal{D}_{\alpha}^{x_i}(p_i) \cup \mathcal{D}_{\alpha}^{y}(©(p_1, \ldots, p_k))$.

$(\mathfrak{R}_{\mathbb{T}}^{\mathfrak{M}\mathcal{D}})$ *for each* $X \notin \mathbb{T}(\mathbf{A})$ *and an arbitrary family* $\{p_x\}_{x \in X}$ *of distinct propositional variables,*

$$\frac{\bigcup_{x \in X} \mathcal{D}_{\mathsf{Y}}^{x}(p_x) \quad \| \quad \bigcup_{x \in X} \mathcal{D}_{\mathsf{N}}^{x}(p_x)}{\bigcup_{x \in X} \mathcal{D}_{\text{⅄}}^{x}(p_x) \quad \| \quad \bigcup_{x \in X} \mathcal{D}_{\text{И}}^{x}(p_x)}$$

Observe that, if the Σ-nd-B-matrix $\mathfrak{M}$ in the definition above is total, the last group of rule schemas, $(\mathfrak{R}_{\mathbb{T}}^{\mathfrak{M}\mathcal{D}})$, can be ignored, as $\mathbb{T}(\mathbf{A}) = \mathsf{Pow}(L_{\Sigma}(P))$. With respect to the size of the produced system when $\mathfrak{M}$ is finite, we have the groups $(\mathfrak{R}_{\exists}^{\mathfrak{M}\mathcal{D}})$ and $(\mathfrak{R}_{\mathbb{T}}^{\mathfrak{M}\mathcal{D}})$ having an amount of rule schemas exponential in the number of values of $\mathfrak{M}$, while the groups $(\mathfrak{R}_{\mathsf{D}}^{\mathfrak{M}\mathcal{D}})$ and $(\mathfrak{R}_{\Sigma}^{\mathfrak{M}\mathcal{D}})$ have a size polynomial in the number of values of $\mathfrak{M}$.

Example 82. *Let us illustrate the above groups of rule schemas with a three-valued nd-*B*-matrix* $\mathfrak{M} := \langle \mathbf{A}, \mathsf{Y}, \mathsf{N} \rangle$ *over a signature containing but one unary connective* $\neg$ *and two binary connectives* $\wedge$ *and* $\rightarrow$*. We let* $A := \{\mathbf{f}, \bot, \mathbf{t}\}$, $\mathsf{Y} := \{\bot, \mathbf{t}\}$ *and* $\mathsf{N} := \{\bot, \mathbf{f}\}$*. Moreover, the interpretations of the said connectives are given by the following truth-tables:*

	$\neg_{\mathbf{A}}$
$\mathbf{f}$	$\mathbf{t}, \bot$
$\bot$	$\varnothing$
$\mathbf{t}$	$\mathbf{f}$

$\wedge_{\mathbf{A}}$	$\mathbf{f}$	$\bot$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\varnothing$	$\mathbf{f}$
$\bot$	$\varnothing$	$\varnothing$	$\varnothing$
$\mathbf{t}$	$\mathbf{f}$	$\varnothing$	$\mathbf{t}, \bot$

$\rightarrow_{\mathbf{A}}$	$\mathbf{f}$	$\bot$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{t}, \bot$	$\varnothing$	$\mathbf{t}, \bot$
$\bot$	$\varnothing$	$\varnothing$	$\varnothing$
$\mathbf{t}$	$\mathbf{f}$	$\varnothing$	$\mathbf{t}, \bot$

Before going into the groups of rule schemas, the axiomatization algorithm requires $\mathfrak{M}$ *to be sufficiently expressive. It turns out that the propositional variable* p *is enough to separate every pair of distinct truth-values, since:*

- $\mathbf{t} \in \mathsf{Y}$ *and* $\mathbf{f} \in \text{⅄}$*;*

- $\bot \in \mathsf{Y}$ *and* $\mathbf{f} \in \text{⅄}$*; and*

- $\bot \in \mathsf{N}$ *and* $\mathbf{t} \in \text{И}$*.*

A discriminator $\mathcal{D}$ *for* $\mathfrak{M}$*, then, is:*

x	$\mathcal{D}^x_{\mathsf{Y}}$	$\mathcal{D}^x_{\text{⅄}}$	$\mathcal{D}^x_{\mathsf{N}}$	$\mathcal{D}^x_{\text{И}}$
$\mathbf{f}$	$\varnothing$	p	$\varnothing$	$\varnothing$
$\bot$	p	$\varnothing$	p	$\varnothing$
$\mathbf{t}$	p	$\varnothing$	$\varnothing$	p

Having all set, we begin with the group of rule schemas $(\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\exists})$*. Let us organize the choices we have in a table, in which each row corresponds to a subset of* $\{\mathbf{f}, \bot, \mathbf{t}\}$ *and the last column indicates the produced rule schemas (we use "—" to indicate that no choice is available or no rule schema is produced):*

X_1	X_0	$(\dot{\mathcal{D}}^{X_1}_{\text{⅄}}, \dot{\mathcal{D}}^{X_1}_{\text{И}})$	$(\dot{\mathcal{D}}^{X_0}_{\mathsf{Y}}, \dot{\mathcal{D}}^{X_0}_{\mathsf{N}})$	$\dfrac{\dot{\mathcal{D}}^{X_1}_{\text{⅄}} \parallel \dot{\mathcal{D}}^{X_1}_{\text{И}}}{\dot{\mathcal{D}}^{X_0}_{\mathsf{Y}} \parallel \dot{\mathcal{D}}^{X_0}_{\mathsf{N}}}$
$\varnothing$	$\{\mathbf{t}, \mathbf{f}, \bot\}$	$(\varnothing, \varnothing)$	—	—
$\{\mathbf{f}\}$	$\{\bot, \mathbf{t}\}$	$(\{p\}, \varnothing)$	$(\{p\}, \{p\})$	$\dfrac{p \parallel}{p \parallel p}$
$\{\mathbf{t}\}$	$\{\mathbf{f}, \bot\}$	$(\varnothing, \{p\})$	—	—
$\{\bot\}$	$\{\mathbf{f}, \mathbf{t}\}$	—	—	—
$\{\mathbf{f}, \bot\}$	$\{\mathbf{t}\}$	—	$(\{p\}, \varnothing)$	—
$\{\mathbf{t}, \bot\}$	$\{\mathbf{f}\}$	—	—	—
$\{\mathbf{f}, \mathbf{t}\}$	$\{\bot\}$	$(\{p\}, \{p\})$	$(\{p\}, \{p\})$	$\dfrac{p \parallel p}{p \parallel p}$
$\{\mathbf{f}, \bot, \mathbf{t}\}$	$\varnothing$	—	$(\varnothing, \varnothing)$	—

Notice that only two rules are produced in $(\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\exists})$*, and that both are instances of* **(O2)***. Now, we produce the rules of the group* $(\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathsf{D}})$*, this time listing in a table the rule schemas produced for each truth-value:*

x	$\dfrac{\mathcal{D}^x_{\mathsf{Y}}(p), \mathsf{p}_{\mathsf{⅄}}(x) \parallel \mathcal{D}^x_{\mathsf{N}}(p)}{\mathcal{D}^x_{\mathsf{⅄}}(p), \mathsf{p}_{\mathsf{Y}}(x) \parallel \mathcal{D}^x_{\mathsf{И}}(p)}$	$\dfrac{\mathcal{D}^x_{\mathsf{Y}}(p) \parallel \mathcal{D}^x_{\mathsf{N}}(p), \mathsf{p}_{\mathsf{И}}(x)}{\mathcal{D}^x_{\mathsf{⅄}}(p) \parallel \mathcal{D}^x_{\mathsf{И}}(p), \mathsf{p}_{\mathsf{N}}(x)}$
$\mathbf{f}$	$\dfrac{p \parallel}{p \parallel}$	$\dfrac{\parallel}{p \parallel p}$
$\bot$	$\dfrac{p \parallel p}{p \parallel}$	$\dfrac{p \parallel p}{\parallel p}$
$\mathbf{t}$	$\dfrac{p \parallel}{p \parallel p}$	$\dfrac{p \parallel p}{\parallel p}$

Note that, with the exception of the second rule for the value $\mathbf{f}$*, all rule schemas are instances of* **(O2)***. As we will see in Proposition 87, we could have foreseen this in view of the chosen discriminator.*

We proceed now to the schemas of $(\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\Sigma})$*, working on the interpretation of each connective, entry by entry. Below we write one table per connective, in which each row represents an entry of the respective truth-table and is identified by the input tuple of the entry in the first column. The second column shows the output of the connective under the input tuple, and the third column shows the complement of this output with respect to the set of all truth-values* $\{\mathbf{f}, \bot, \mathbf{t}\}$*. Recall from the definition of the schemas in* $(\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\Sigma})$ *that one schema is produced per each value outside the output of the input tuple per entry per connective (that is, those values listed in the third column of the tables below).*

(x)	$\neg_{\mathbf{A}}(x)$	$A \backslash \neg_{\mathbf{A}}(x)$	Rule schemas
$(\mathbf{f})$	$\{\bot,\mathbf{t}\}$	$\{\mathbf{f}\}$	$\dfrac{\parallel}{p, \neg p \parallel}\,\neg^{\mathbf{f}}_{\mathbf{f}}$
$(\bot)$	$\varnothing$	$\{\mathbf{f}, \bot, \mathbf{t}\}$	$\dfrac{p \parallel p}{\neg p \parallel}\,\neg^{\bot}_{\mathbf{f}}$ $\quad \dfrac{p, \neg p \parallel p, \neg p}{\parallel}\,\neg^{\bot}_{\bot}$ $\quad \dfrac{p, \neg p \parallel p}{\parallel \neg p}\,\neg^{\bot}_{\mathbf{t}}$
$(\mathbf{t})$	$\{\mathbf{f}\}$	$\{\bot, \mathbf{t}\}$	$\dfrac{p, \neg p \parallel \neg p}{\parallel p}\,\neg^{\mathbf{t}}_{\bot}$ $\quad \dfrac{p, \neg p \parallel}{\parallel p, \neg p}\,\neg^{\mathbf{t}}_{\mathbf{t}}$

(x,y)	$\wedge_{\mathbf{A}}(x,y)$	$A\backslash\wedge_{\mathbf{A}}(x,y)$	Rule schemas
$(\mathbf{f},\mathbf{f})$	$\{\mathbf{f}\}$	$\{\bot,\mathbf{t}\}$	$\frac{p\wedge q \parallel p\wedge q}{p,q \parallel}\wedge^{\mathbf{ff}}_{\bot}$ $\quad \frac{p\wedge q \parallel}{p,q \parallel p\wedge q}\wedge^{\mathbf{ff}}_{\mathbf{t}}$
$(\mathbf{f},\bot)$	$\varnothing$	$\{\mathbf{f},\bot,\mathbf{t}\}$	$\frac{q \parallel q}{p,p\wedge q \parallel}\wedge^{\mathbf{f}\bot}_{\mathbf{f}}$ $\quad \frac{q,p\wedge q \parallel q,p\wedge q}{p \parallel}\wedge^{\mathbf{f}\bot}_{\bot}$ $\quad \frac{q,p\wedge q \parallel q}{p \parallel p\wedge q}\wedge^{\mathbf{f}\bot}_{\mathbf{t}}$
$(\mathbf{f},\mathbf{t})$	$\{\mathbf{f}\}$	$\{\bot,\mathbf{t}\}$	$\frac{q,p\wedge q \parallel p\wedge q}{p \parallel q}\wedge^{\mathbf{ft}}_{\bot}$ $\quad \frac{q,p\wedge q \parallel}{p \parallel q,p\wedge q}\wedge^{\mathbf{ft}}_{\mathbf{t}}$
$(\bot,\mathbf{f})$	$\varnothing$	$\{\mathbf{f},\bot,\mathbf{t}\}$	$\frac{p \parallel p}{q,p\wedge q \parallel}\wedge^{\bot\mathbf{f}}_{\mathbf{f}}$ $\quad \frac{p,p\wedge q \parallel p,p\wedge q}{q \parallel}\wedge^{\bot\mathbf{f}}_{\bot}$ $\quad \frac{p,p\wedge q \parallel p}{q \parallel p\wedge q}\wedge^{\bot\mathbf{f}}_{\mathbf{t}}$
$(\bot,\bot)$	$\varnothing$	$\{\mathbf{f},\bot,\mathbf{t}\}$	$\frac{p,q \parallel p,q}{p\wedge q \parallel}\wedge^{\bot\bot}_{\mathbf{f}}$ $\quad \frac{p,q,p\wedge q \parallel p,q,p\wedge q}{\parallel}\wedge^{\bot\bot}_{\bot}$ $\quad \frac{p,q,p\wedge q \parallel p,q}{\parallel p\wedge q}\wedge^{\bot\bot}_{\mathbf{t}}$
$(\bot,\mathbf{t})$	$\varnothing$	$\{\mathbf{f},\bot,\mathbf{t}\}$	$\frac{p,q \parallel}{p\wedge q \parallel q}\wedge^{\bot\mathbf{t}}_{\mathbf{f}}$ $\quad \frac{p,q,p\wedge q \parallel p\wedge q}{\parallel q}\wedge^{\bot\mathbf{t}}_{\bot}$ $\quad \frac{p,q,p\wedge q \parallel}{\parallel q,p\wedge q}\wedge^{\bot\mathbf{t}}_{\mathbf{t}}$
$(\mathbf{t},\mathbf{f})$	$\{\mathbf{f}\}$	$\{\bot,\mathbf{t}\}$	$\frac{p,p\wedge q \parallel p\wedge q}{q \parallel p}\wedge^{\mathbf{tf}}_{\bot}$ $\quad \frac{p,p\wedge q \parallel}{q \parallel p,p\wedge q}\wedge^{\mathbf{tf}}_{\mathbf{t}}$
$(\mathbf{t},\bot)$	$\varnothing$	$\{\mathbf{f},\bot,\mathbf{t}\}$	$\frac{p,q \parallel q}{p\wedge q \parallel p}\wedge^{\mathbf{t}\bot}_{\mathbf{f}}$ $\quad \frac{p,q,p\wedge q \parallel q,p\wedge q}{\parallel p}\wedge^{\mathbf{t}\bot}_{\bot}$ $\quad \frac{p,q,p\wedge q \parallel q}{\parallel p,p\wedge q}\wedge^{\mathbf{t}\bot}_{\mathbf{t}}$
$(\mathbf{t},\mathbf{t})$	$\{\bot,\mathbf{t}\}$	$\{\mathbf{f}\}$	$\frac{p,q \parallel}{p\wedge q \parallel p,q}\wedge^{\mathbf{tt}}_{\mathbf{f}}$

(x, y)	$\to_{\mathbf{A}}(x, y)$	$A \backslash \to_{\mathbf{A}}(x, y)$	Rule schemas
$(\mathbf{f},\mathbf{f})$	$\{\bot, \mathbf{t}\}$	$\{\mathbf{f}\}$	$\dfrac{\;\parallel\;}{p,q,p\to q \parallel}\to^{\mathbf{ff}}_{\mathbf{f}}$
$(\mathbf{f},\bot)$	$\varnothing$	$\{\mathbf{f}, \bot, \mathbf{t}\}$	$\dfrac{q \parallel q}{p,p\to q \parallel}\to^{\mathbf{f}\bot}_{\mathbf{f}}$ $\dfrac{q,p\to q \parallel q,p\to q}{p \parallel}\to^{\mathbf{f}\bot}_{\bot}$ $\dfrac{q,p\to q \parallel q}{p \parallel p\to q}\to^{\mathbf{f}\bot}_{\mathbf{t}}$
$(\mathbf{f},\mathbf{t})$	$\{\bot, \mathbf{t}\}$	$\{\mathbf{f}\}$	$\dfrac{q \parallel}{p,p\to q \parallel q}\to^{\mathbf{ft}}_{\mathbf{f}}$
$(\bot,\mathbf{f})$	$\varnothing$	$\{\mathbf{f}, \bot, \mathbf{t}\}$	$\dfrac{p \parallel p}{q,p\to q \parallel}\to^{\bot\mathbf{f}}_{\mathbf{f}}$ $\dfrac{p,p\to q \parallel p,p\to q}{q \parallel}\to^{\bot\mathbf{f}}_{\bot}$ $\dfrac{p,p\to q \parallel p}{q \parallel p\to q}\to^{\bot\mathbf{f}}_{\mathbf{t}}$
$(\bot,\bot)$	$\varnothing$	$\{\mathbf{f}, \bot, \mathbf{t}\}$	$\dfrac{p,q \parallel p,q}{p\to q \parallel}\to^{\bot\bot}_{\mathbf{f}}$ $\dfrac{p,q,p\to q \parallel p,q,p\to q}{\parallel}\to^{\bot\bot}_{\bot}$ $\dfrac{p,q,p\to q \parallel p,q}{\parallel p\to q}\to^{\bot\bot}_{\mathbf{t}}$
$(\bot,\mathbf{t})$	$\varnothing$	$\{\mathbf{f}, \bot, \mathbf{t}\}$	$\dfrac{p,q \parallel p}{p\to q \parallel q}\to^{\bot\mathbf{t}}_{\mathbf{f}}$ $\dfrac{p,q,p\to q \parallel p,p\to q}{\parallel q}\to^{\bot\mathbf{t}}_{\bot}$ $\dfrac{p,q,p\to q \parallel p}{\parallel q,p\to q}\to^{\bot\mathbf{t}}_{\mathbf{t}}$
$(\mathbf{t},\mathbf{f})$	$\{\mathbf{f}\}$	$\{\bot, \mathbf{t}\}$	$\dfrac{p,p\to q \parallel p\to q}{q \parallel p}\to^{\mathbf{tf}}_{\bot}$ $\dfrac{p,p\to q \parallel}{q \parallel p,p\to q}\to^{\mathbf{tf}}_{\mathbf{t}}$
$(\mathbf{t},\bot)$	$\varnothing$	$\{\mathbf{f}, \bot, \mathbf{t}\}$	$\dfrac{p,q \parallel q}{p\to q \parallel p}\to^{\mathbf{t}\bot}_{\mathbf{f}}$ $\dfrac{p,q,p\to q \parallel q,p\to q}{\parallel p}\to^{\mathbf{t}\bot}_{\bot}$ $\dfrac{p,q,p\to q \parallel q}{\parallel p,p\to q}\to^{\mathbf{t}\bot}_{\mathbf{t}}$
$(\mathbf{t},\mathbf{t})$	$\{\bot, \mathbf{t}\}$	$\{\mathbf{f}\}$	$\dfrac{p,q \parallel}{p\to q \parallel p,q}\to^{\mathbf{tt}}_{\mathbf{f}}$

Finally, we produce the schemas of the group ($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathbb{T}}$)*, which must be nonempty, since there are empty outputs in some entries of the interpretations of* $\mathfrak{M}$*. We note that any subset of* $\{\mathbf{f}, \bot, \mathbf{t}\}$ *containing the value* $\bot$ *induce a nontotal subcomponent of* $\mathfrak{M}$*. In other words, we have* $\mathbb{T}(\mathbf{A}) = \{\varnothing, \{\mathbf{f}\}, \{\mathbf{t}\}, \{\mathbf{f}, \mathbf{t}\}\}$*. Let* $p := p_{\mathbf{f}}$*,* $q := p_{\bot}$ *and* $r := p_{\mathbf{t}}$*. We list the subsets not in* $\mathbb{T}(\mathbf{A})$ *in the table below, together with the corresponding rule schemas:*

$X \notin \mathbb{T}(\mathbf{A})$	$\dfrac{\bigcup_{x\in X} \mathcal{D}^x_{\mathsf{Y}}(p_x) \parallel \bigcup_{x\in X} \mathcal{D}^x_{\mathsf{N}}(p_x)}{\bigcup_{x\in X} \mathcal{D}^x_{\mathsf{⅄}}(p_x) \parallel \bigcup_{x\in X} \mathcal{D}^x_{\mathsf{И}}(p_x)}$
$\{\bot\}$	$\dfrac{q \parallel q}{\parallel}$
$\{\mathbf{f}, \bot\}$	$\dfrac{q \parallel q}{p \parallel}$
$\{\mathbf{t}, \bot\}$	$\dfrac{q,r \parallel q}{\parallel r}$
$\{\mathbf{f}, \bot, \mathbf{t}\}$	$\dfrac{q,r \parallel q}{p \parallel r}$

The calculus $\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$, then, consists of all rule schemas in the last column of the above tables.

The next theorem proves that the system $\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$ described in Definition 81 is sound for $\mathfrak{M}$.

Theorem 83. *If $\mathcal{D}$ is a discriminator for a Σ-nd-B-matrix $\mathfrak{M} := \langle \mathbf{A}, \mathsf{Y}, \mathsf{N}\rangle$, then the* SET^2-SET^2 *calculus $\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$ is sound with respect to $\mathfrak{M}$.*

Proof. We will show that any $\mathfrak{M}$-valuation that constituted a countermodel for a schema of $\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$ would lead to an absurd. The argument will cover each of the groups of rule schemas of the concerned calculus.

($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\exists}$) Consider a schema $\mathfrak{s} := \dfrac{\dot{\mathcal{D}}^{X_1}_{\mathsf{⅄}} \parallel \dot{\mathcal{D}}^{X_1}_{\mathsf{И}}}{\dot{\mathcal{D}}^{X_0}_{\mathsf{Y}} \parallel \dot{\mathcal{D}}^{X_0}_{\mathsf{N}}}$, for some $X_1 \subseteq A$ and some choice of $(\dot{\mathcal{D}}^{X_0}_{\mathsf{Y}}, \dot{\mathcal{D}}^{X_0}_{\mathsf{N}})$ and $(\dot{\mathcal{D}}^{X_1}_{\mathsf{⅄}}, \dot{\mathcal{D}}^{X_1}_{\mathsf{И}})$. Suppose that $\mathfrak{s}$ does not hold in $\mathfrak{M}$, with the $\mathfrak{M}$-valuation v witnessing this fact. We will prove that, given a propositional variable p, $v(p) \neq x$, for all $x \in A$, an absurd. For that purpose, let $x \in A$. In case $x \in X_1$, there must be a separator S in $\mathcal{D}^x_{\tilde{\alpha}}$, for some $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$, such that $v(\mathrm{S}(p)) \in \alpha$. By Lemma 79, this implies that $v(p) \neq x$. The reasoning is similar in case $x \in X_0$.

($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathsf{D}}$) Let $p \in P$ and $x \in A$ be such that $x \in \alpha$, with $\alpha \in \{\mathsf{Y}, \mathsf{⅄}\}$. Suppose that there is an $\mathfrak{M}$-valuation v under which the schema $\dfrac{\mathcal{D}^x_{\mathsf{Y}}(p), \mathsf{p}_{\mathsf{⅄}}(x) \parallel \mathcal{D}^x_{\mathsf{N}}(p)}{\mathcal{D}^x_{\mathsf{⅄}}(p), \mathsf{p}_{\mathsf{Y}}(x) \parallel \mathcal{D}^x_{\mathsf{И}}(p)}$ does not hold. Then, $v(\mathsf{p}_\alpha(x)) \subseteq \tilde{\alpha}$, and thus, since $x \in \alpha$, it follows that $v(p) \in \tilde{\alpha}$. On the other hand, since $v[\mathcal{D}^x_\alpha(p)] \subseteq \alpha$ and $v[\mathcal{D}^x_{\tilde{\alpha}}(p)] \subseteq \tilde{\alpha}$ for each $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$, by Lemma 79 we have $v(p) = x$, a contradiction. The proof for the other schema is analogous.

$(\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\Sigma})$ Let $© \in \Sigma_k$, $X := \{x_i\}_{i=1}^{k}$ be a family of truth-values of $\mathfrak{M}$, $y \notin ©_{\mathbf{A}}(x_1, \ldots, x_k)$ and let $(p_1, \ldots, p_k)$ be a sequence of distinct propositional variables. Suppose the schema $\frac{\Theta^{©,X,y}_{\mathsf{Y}} \parallel \Theta^{©,X,y}_{\mathsf{N}}}{\Theta^{©,X,y}_{\rotatebox{180}{Y}} \parallel \Theta^{©,X,y}_{\reflectbox{N}}}$ does not hold under v. By Lemma 79, we have that $v(p_i) = x_i$, for all $1 \leq i \leq k$, and $v(©(p_1, \ldots, p_k)) = y$. It follows that $y = v(©(p_1, \ldots, p_k)) \in ©_{\mathbf{A}}(v(p_1), \ldots, v(p_k)) = ©_{\mathbf{A}}(x_1, \ldots, x_k)$, contradicting one of the assumptions.

$(\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathbb{T}})$ Assume that $X \notin \mathbb{T}(\mathbf{A})$ and let $\{p_x\}_{x \in X}$ be a family of distinct propositional variables. If the schema $\frac{\bigcup_{x \in X} \mathcal{D}^{x}_{\mathsf{Y}}(p_x) \parallel \bigcup_{x \in X} \mathcal{D}^{x}_{\mathsf{N}}(p_x)}{\bigcup_{x \in X} \mathcal{D}^{x}_{\rotatebox{180}{Y}}(p_x) \parallel \bigcup_{x \in X} \mathcal{D}^{x}_{\reflectbox{N}}(p_x)}$ does not hold under an $\mathfrak{M}$-valuation v, then Lemma 79 guarantees that $v(p_x) = x$ for each $x \in X$. Hence, since $p_x \in P$ for each $x \in X$, we have $X \subseteq v[L_\Sigma(P)] \in \mathbb{T}(\mathbf{A})$, contradicting the assumption.

$\square$

In what follows, we fix a sufficiently expressive Σ-nd-B-matrix $\mathfrak{M} := \langle \mathbf{A}, \mathsf{Y}, \mathsf{N} \rangle$, a discriminator $\mathcal{D}$ for $\mathfrak{M}$ and treat each of the groups of rule schemas $(\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\exists})$, $(\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathsf{D}})$, $(\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\Sigma})$ and $(\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathbb{T}})$ as $\textsc{Set}^2$-$\textsc{Set}^2$ systems by themselves, that is, each one inducing its own B-consequence relation. We proceed now to prove that $\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$ is complete for $\mathfrak{M}$ and $\mathcal{D}^{\bowtie}$-analytic. In this direction, we shall make use of Lemma 84 below, which contains four items, each one, once again, referring to a group of schemas of $\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$. The statement seems complicated, but its formulation is oriented in order to facilitate the completeness theorem presented right after. Intuitively, given a B-statement $\mathfrak{s}$ and assuming that there is no $\mathcal{D}^{\bowtie}$-analytic proof of it in $\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$, items (1) and (2) give us the resources to define a mapping $f : \mathsf{subf}(\mathfrak{s}) \to A$ that, by items (3) and (4), can be extended, via the property of effectiveness of Σ-nd-B-matrices, to a countermodel for $\mathfrak{s}$ in $\mathfrak{M}$.

Lemma 84. *For all* B*-statements* $\mathfrak{s}$ *of the form* $\left(\begin{array}{c|c} \Omega^{\mathsf{c}}_{\reflectbox{S}} & \Omega^{\mathsf{c}}_{\mathsf{S}} \\ \hline \Omega_{\mathsf{S}} & \Omega_{\reflectbox{S}} \end{array}\right)$*:*

1. *if* $\frac{\Omega^{\mathsf{c}}_{\reflectbox{S}}}{\Omega_{\mathsf{S}}} \nmid \frac{\Omega^{\mathsf{c}}_{\mathsf{S}}}{\Omega_{\reflectbox{S}}}{}^{\mathcal{D}^{\bowtie}}_{\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\exists}}$*, then, for all* $\varphi \in \mathsf{subf}(\mathfrak{s})$*, there is an* $x \in A$ *such that* $\mathcal{D}^{x}_{\alpha}(\varphi) \subseteq \Omega_\beta$ *and* $\mathcal{D}^{x}_{\tilde{\alpha}}(\varphi) \subseteq \Omega^{\mathsf{c}}_\beta$*, for each* $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \reflectbox{S})\}$*;*
2. *if* $\frac{\Omega^{\mathsf{c}}_{\reflectbox{S}}}{\Omega_{\mathsf{S}}} \nmid \frac{\Omega^{\mathsf{c}}_{\mathsf{S}}}{\Omega_{\reflectbox{S}}}{}^{\mathcal{D}^{\bowtie}}_{\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathsf{D}}}$*, then, for every* $\varphi \in \mathsf{subf}(\mathfrak{s})$ *and* $x \in A$ *such that* $\mathcal{D}^{x}_{\alpha}(\varphi) \subseteq \Omega_\beta$ *and* $\mathcal{D}^{x}_{\tilde{\alpha}}(\varphi) \subseteq \Omega^{\mathsf{c}}_\beta$*, we have* $x \in \alpha$ *iff* $\varphi \in \Omega_\beta$*, for each* $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \reflectbox{S})\}$*;*

3. *if* $\frac{\Omega^{\mathsf{c}}_{\mathsf{Ƨ}}}{\Omega_{\mathsf{S}}} \nmid \frac{\Omega^{\mathsf{c}}_{\mathsf{S}}}{\Omega_{\mathsf{Ƨ}}} \overset{\mathcal{D}^{\bowtie}}{\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\Sigma}}$, *then for every* $© \in \Sigma_k$, $\varphi := ©(\varphi_1, \ldots, \varphi_k) \in \mathsf{subf}(\mathfrak{s})$ *and* $x_1, \ldots, x_k \in A$ *with* $\mathcal{D}^{x_i}_{\alpha}(\varphi_i) \subseteq \Omega_\beta$ *and* $\mathcal{D}^{x_i}_{\tilde{\alpha}}(\varphi_i) \subseteq \Omega^{\mathsf{c}}_\beta$, *for each* $1 \leq i \leq k$ *and* $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$, *we have that* $\mathcal{D}^{y}_{\alpha}(\varphi) \subseteq \Omega_\beta$ *and* $\mathcal{D}^{y}_{\tilde{\alpha}}(\varphi) \subseteq \Omega^{\mathsf{c}}_\beta$ *for each* $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$ *implies* $y \in ©_{\mathbf{A}}(x_1, \ldots, x_k)$*;*
4. *if* $\frac{\Omega^{\mathsf{c}}_{\mathsf{Ƨ}}}{\Omega_{\mathsf{S}}} \nmid \frac{\Omega^{\mathsf{c}}_{\mathsf{S}}}{\Omega_{\mathsf{Ƨ}}} \overset{\mathcal{D}^{\bowtie}}{\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathbb{T}}}$, *then* $\{x \in A \mid \mathcal{D}^{x}_{\alpha}(\varphi) \subseteq \Omega_\beta$ *and* $\mathcal{D}^{x}_{\tilde{\alpha}}(\varphi) \subseteq \Omega^{\mathsf{c}}_\beta$,
for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$ *and* $\varphi \in \mathsf{subf}(\mathfrak{s})\} \in \mathbb{T}(\mathbf{A})$.

Proof. We prove below the contrapositive version of each item.

1. Assume that for some $\varphi \in \mathsf{subf}(\mathfrak{s})$ there is no $x \in A$ such that $\mathcal{D}^{x}_{\alpha}(\varphi) \subseteq \Omega_\beta$ and $\mathcal{D}^{x}_{\tilde{\alpha}}(\varphi) \subseteq \Omega^{\mathsf{c}}_\beta$, for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$. Consider then the set

$$X_{\mathsf{1}} := \{x \in A \mid \mathcal{D}^{x}_{\mathsf{⅄}} \cap \Omega_{\mathsf{S}} \neq \varnothing \text{ or } \mathcal{D}^{x}_{\mathsf{И}} \cap \Omega_{\mathsf{Ƨ}} \neq \varnothing\}$$

   and let $X_{\mathsf{0}} := A \backslash X_{\mathsf{1}}$. Define, for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$, the set $\dot{\mathcal{D}}^{X_{\mathsf{1}}}_{\tilde{\alpha}}$ by choosing, for each $x \in X_{\mathsf{1}}$, a formula S such that $\mathrm{S}(\varphi) \in \mathcal{D}^{x}_{\tilde{\alpha}} \cap \Omega_\beta$, when present. Similarly, define the set $\dot{\mathcal{D}}^{X_{\mathsf{0}}}_{\alpha}$ by choosing, for each $x \in X_{\mathsf{0}}$, a formula S such that $\mathcal{D}^{x}_{\alpha}(\varphi) \cap \Omega^{\mathsf{c}}_\beta$, when present. Notice that the construction of X_{1} guarantees the existence of the pairs $(\dot{\mathcal{D}}^{X_{\mathsf{0}}}_{\mathsf{Y}}, \dot{\mathcal{D}}^{X_{\mathsf{0}}}_{\mathsf{N}})$ and $(\dot{\mathcal{D}}^{X_{\mathsf{1}}}_{\mathsf{⅄}}, \dot{\mathcal{D}}^{X_{\mathsf{1}}}_{\mathsf{И}})$. Since $\dot{\mathcal{D}}^{X_{\mathsf{0}}}_{\alpha}(\varphi) \subseteq \Omega^{\mathsf{c}}_\beta \cap \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$ and $\dot{\mathcal{D}}^{X_{\mathsf{1}}}_{\tilde{\alpha}}(\varphi) \subseteq \Omega_\beta \cap \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$ for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$, we have $\frac{\Omega^{\mathsf{c}}_{\mathsf{Ƨ}}}{\Omega_{\mathsf{S}}} \mid \frac{\Omega^{\mathsf{c}}_{\mathsf{S}}}{\Omega_{\mathsf{Ƨ}}} \overset{\mathcal{D}^{\bowtie}}{\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\exists}}$.
2. Suppose that, for some $\varphi \in \mathsf{subf}(\mathfrak{s})$ and $x \in A$ such that $\mathcal{D}^{x}_{\alpha}(\varphi) \subseteq \Omega_\beta$ and $\mathcal{D}^{x}_{\tilde{\alpha}}(\varphi) \subseteq \Omega^{\mathsf{c}}_\beta$ for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$, we have either (a): $x \in \gamma$ and $\varphi \notin \Omega_\delta$ (i.e. $\varphi \in \Omega^{\mathsf{c}}_\delta$) or (b): $x \notin \gamma$ and $\varphi \in \Omega_\gamma$, for some $(\gamma, \delta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$. Notice that, for any $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$, we have $\mathcal{D}^{x}_{\alpha}(\varphi) \subseteq \Omega_\beta \cap \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$ and $\mathcal{D}^{x}_{\tilde{\alpha}}(\varphi) \subseteq \Omega^{\mathsf{c}}_\beta \cap \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$, implying that, in any of the cases (a) or (b), we have $(\mathcal{D}^{x}_{\gamma}(p) \cup \mathsf{p}_\gamma(p))(\varphi) \subseteq \Omega_\delta \cap \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$ and $(\mathcal{D}^{x}_{\tilde{\gamma}}(p) \cup \mathsf{p}_{\tilde{\gamma}}(p))(\varphi) \subseteq \Omega^{\mathsf{c}}_\delta \cap \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$. Thus, we have $\frac{\Omega^{\mathsf{c}}_{\mathsf{Ƨ}}}{\Omega_{\mathsf{S}}} \mid \frac{\Omega^{\mathsf{c}}_{\mathsf{S}}}{\Omega_{\mathsf{Ƨ}}} \overset{\mathcal{D}^{\bowtie}}{\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathsf{D}}}$, by an instance of one of the schemas of $(\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathsf{D}})$ and property **(D2)**.

3. Suppose there is a connective $© \in \Sigma_k$, a formula $\varphi = ©(\varphi_1, \ldots, \varphi_k) \in \mathsf{subf}(\mathfrak{s})$, a sequence $(x_1, \ldots, x_k)$ of truth-values with $\mathcal{D}^{x_i}_{\alpha}(\varphi_i) \subseteq \Omega_\beta$ and $\mathcal{D}^{x_i}_{\tilde{\alpha}}(\varphi_i) \subseteq \Omega^{\mathsf{c}}_\beta$, for each $1 \leq i \leq k$ and $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$, and some $y \notin ©_{\mathbf{A}}(x_1, \ldots, x_k)$ such that $\mathcal{D}^{y}_{\alpha}(\varphi) \subseteq \Omega_\beta$ and $\mathcal{D}^{y}_{\tilde{\alpha}}(\varphi) \subseteq \Omega^{\mathsf{c}}_\beta$ for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$. Then $\bigcup_{i=1}^{k} \mathcal{D}^{x_i}_{\alpha}(\varphi_i) \cup \mathcal{D}^{y}_{\alpha}(\varphi) \subseteq \Omega_\beta \cap \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$ and $\bigcup_{i=1}^{k} \mathcal{D}^{x_i}_{\tilde{\alpha}}(\varphi_i) \cup \mathcal{D}^{y}_{\tilde{\alpha}}(\varphi) \subseteq \Omega^{\mathsf{c}}_\beta \cap \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$ for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$, and thus we have $\frac{\Omega^{\mathsf{c}}_{\mathsf{Ƨ}}}{\Omega_{\mathsf{S}}} \Big| \frac{\Omega^{\mathsf{c}}_{\mathsf{S}}}{\Omega_{\mathsf{Ƨ}}} \overset{\mathcal{D}^{\bowtie}}{\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\Sigma}}$.
4. Let $X := \{x \in A \mid \mathcal{D}^{x}_{\alpha}(\varphi) \subseteq \Omega_\beta$ and $\mathcal{D}^{x}_{\tilde{\alpha}}(\varphi) \subseteq \Omega^{\mathsf{c}}_\beta,$ for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$ and $\varphi \in \mathsf{subf}(\mathfrak{s})\}$. For each $x \in X$, pick a formula $\varphi_x \in \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$ such that $\mathcal{D}^{x}_{\alpha}(\varphi_x) \subseteq \Omega_\beta$ and $\mathcal{D}^{x}_{\tilde{\alpha}}(\varphi_x) \subseteq \Omega^{\mathsf{c}}_\beta$, for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$. Easily, then, for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$, we have $\bigcup_{x \in X} \mathcal{D}^{x}_{\alpha}(\varphi) \subseteq \Omega_\beta \cap \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$ and $\bigcup_{x \in X} \mathcal{D}^{x}_{\tilde{\alpha}}(\varphi) \subseteq \Omega^{\mathsf{c}}_\beta \cap \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$, and so $\frac{\Omega^{\mathsf{c}}_{\mathsf{Ƨ}}}{\Omega_{\mathsf{S}}} \Big| \frac{\Omega^{\mathsf{c}}_{\mathsf{S}}}{\Omega_{\mathsf{Ƨ}}} \overset{\mathcal{D}^{\bowtie}}{\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathbb{T}}}$ if $X \notin \mathbb{T}(\mathbf{A})$.

□

Theorem 85. *The* Set^2-Set^2 *system* $\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$ *is complete for* $\mathfrak{M}$ *and* $\mathcal{D}^{\bowtie}$*-analytic.*

Proof. Let $\mathfrak{s} := \left(\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}} \Big| \frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}\right)$ be a B-statement and suppose that (a): $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}} \nmid \frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}} \overset{\mathcal{D}^{\bowtie}}{\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}}$. Our goal is to build an $\mathfrak{M}$-valuation witnessing that $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}} \nmid \frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}} \mathfrak{M}$. From (a), by **(C2)**, we have that (b): there are $\Phi_{\mathsf{Y}} \subseteq \Omega_{\mathsf{S}} \subseteq \Phi^{\mathsf{c}}_{\mathsf{⅄}}$ and $\Phi_{\mathsf{N}} \subseteq \Omega_{\mathsf{Ƨ}} \subseteq \Phi^{\mathsf{c}}_{\mathsf{И}}$ such that $\frac{\Omega^{\mathsf{c}}_{\mathsf{Ƨ}}}{\Omega_{\mathsf{S}}} \nmid \frac{\Omega^{\mathsf{c}}_{\mathsf{S}}}{\Omega_{\mathsf{Ƨ}}} \overset{\mathcal{D}^{\bowtie}}{\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}}$. Consider then a mapping $f : \mathsf{subf}(\mathfrak{s}) \to A$ with (c): $f(\varphi) \in \alpha$ iff $\varphi \in \Omega_\beta$, for $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \mathsf{Ƨ})\}$, whose existence is guaranteed by items (1) and (2) of Lemma 84. Notice that items (3) and (4) of this same lemma imply, respectively, that $f(©(\varphi_1, \ldots, \varphi_k)) \in ©_{\mathbf{A}}(f(\varphi_1), \ldots, f(\varphi_k))$ for every $©(\varphi_1, \ldots, \varphi_k) \in \mathsf{subf}(\mathfrak{s})$, and $f(\mathsf{subf}(\mathfrak{s})) \in \mathbb{T}(\mathbf{A})$. Hence, f may be extended to an $\mathfrak{M}$-valuation v, and, from (b) and (c), we have $v(\Phi_\alpha) \subseteq \alpha$ for each $\alpha \in \{\mathsf{Y}, \mathsf{N}, \mathsf{⅄}, \mathsf{И}\}$, so $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}} \nmid \frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}} \mathfrak{M}$. □

The proof-search algorithm described in Section 4.3 makes the axiomatization procedure just presented even more attractive, since this procedure delivers a $\mathcal{D}^{\bowtie}$–analytic calculus for any finite and sufficiently expressive Σ-nd-B-matrix $\mathfrak{M}$, where $\mathcal{D}$ is a discriminator for $\mathfrak{M}$. It follows then that Proof-Search is a proof-search algorithm for such axiomatization running in at most exponential time in the size of the B-statement of interest. For experimenting with the axiomatization procedure and searching for proofs over the generated calculus, one can make use of the implementation that may be found at `https://github.com/greati/logicantsy`. See Appendix A for detailed instructions. We should also emphasize that, by Theorem 74 and the axiomatization procedure given in Definition 81, we have:

Corollary 86. *Any finite and sufficiently expressive Σ-nd-B-matrix $\mathfrak{M}$ whose induced axiomatization contains only rules with at most one succedent is decidable in polynomial time.*

From this result, we obtain immediately that, in view of the finite and analytic $\textsc{Set}^2$-$\textsc{Set}^2$ system presented in Example 56, which was produced using the axiomatization algorithm defined and studied in the present section, the logic of information sources introduced in [4] and presented in Example 17 is decidable in polynomial time. After noticing this fact, we should observe that, actually, this result could already have been obtained in the original, one-dimensional presentation of this logic, in view of the algorithm for axiomatizing in $\textsc{Set}$-$\textsc{Set}$ sufficiently expressive matrices [43]. We have no reasons to believe, however, that this complexity result should be preserved in general when passing from two to one dimension.

5.3. Simplifying the axiomatization

The axiomatization presented in Definition 81 can be considerably simplified by some basic streamlining procedures. Before introducing them, we provide a simple result establishing a sufficient condition for the group of rule schemas ($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathsf{D}}$) to contain only instances of overlap. We will see in a moment that, in this case, this group may be ignored.

Proposition 87. *Let $\mathcal{D}$ be a discriminator for a sufficiently expressive Σ-nd-B-matrix $\mathfrak{M} := \langle \mathbf{A}, \mathsf{Y}, \mathsf{N}\rangle$ and $p \in P$. For all $\alpha \in \{\mathsf{Y}, \mathsf{N}, \curlywedge, \text{И}\}$, if $p \in \mathcal{D}^x_\alpha(p)$ whenever $x \in \alpha$, then the schemas in ($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathsf{D}}$) are all instances of property* **(O2)**.

Proof. Suppose that (a): for all $\alpha \in \{\mathsf{Y}, \mathsf{N}, \curlywedge, \text{И}\}$, if $x \in \alpha$, then $p \in \mathcal{D}^x_\alpha(p)$. Let us consider the schema $\frac{\mathcal{D}^x_{\mathsf{Y}}(p), \mathsf{p}_{\curlywedge}(x) \,\|\, \mathcal{D}^x_{\mathsf{N}}(p)}{\mathcal{D}^x_{\curlywedge}(p), \mathsf{p}_{\mathsf{Y}}(x) \,\|\, \mathcal{D}^x_{\text{И}}(p)}$ and the proof for the other schema will be analogous. By cases, suppose that (b): $x \in \mathsf{Y}$. Then, $\mathsf{p}_{\mathsf{Y}}(x) = \{p\}$, and, by (a) and (b), we have $p \in \mathcal{D}^x_{\mathsf{Y}}(p)$, thus the schema under concern is an instance of **(O2)**. The case $x \in \curlywedge$ is similar. □

Example 88. *The discriminators presented in Example 78 satisfy the precondition of the previous proposition, thus, for each of them, the schemas in ($\mathfrak{R}^{\mathfrak{M}\mathcal{D}}_{\mathsf{D}}$) are all instances of property* **(O2)**.

We now define three simple streamlining operations for finite and finitary Set^2-Set^2 systems and then prove that they preserve the induced B-consequence, that is, they preserve adequacy. Moreover, we show that, when the input is Θ-analytic, the result is also Θ-analytic. It will also be made clear that, in view of the finitariness and finiteness hypotheses, such procedures actually describe algorithms having straightforward implementations.

Definition 89. *Given two rule instances $\frac{\Phi_{\mathsf{Y}} \,\|\, \Phi_{\mathsf{N}}}{\Phi_{\curlywedge} \,\|\, \Phi_{\text{И}}}$ and $\frac{\Psi_{\mathsf{Y}} \,\|\, \Psi_{\mathsf{N}}}{\Psi_{\curlywedge} \,\|\, \Psi_{\text{И}}}$, we say that the latter is a* dilution *of the former in case $\Phi_\alpha \subseteq \Psi_\alpha$ for each $\alpha \in \{\mathsf{Y}, \curlywedge, \mathsf{N}, \text{И}\}$. Equivalently, we may say that the former is a* subinstance *of the latter. This same terminology applies to rule schemas (using, in this case,* subschema *instead of subinstance). Given two rules of inference R_1, R_2, we write $R_1 \geq R_2$ whenever every rule instance in R_1 is a dilution of an instance of R_2.*

Definition 90. *Let $\mathfrak{R}$ be a finite and finitary* Set^2-Set^2 *system.*

- *Let $O(\mathfrak{R})$ be the system resulting from removing from the rules of inference of $\mathfrak{R}$ all rules of inference whose instances are all cases of* **(O2)**.

- *Consider an enumeration* $R_1, \ldots, R_n$ *of the rules of inference of* $\mathfrak{R}$*. Let* $D(\mathfrak{R}) := D^n(\mathfrak{R})$*, where*

$$D^1(\mathfrak{R}) := \mathfrak{R}$$
$$D^{i+1}(\mathfrak{R}) := \begin{cases} D^i(\mathfrak{R})\backslash\{R_i\} & \textit{if } R_i \geq R_j, \textit{ for some } R_j \in D^i(\mathfrak{R}) \\ D^i(\mathfrak{R}) & \textit{otherwise} \end{cases}$$

In short, operation D *removes the rules of inference of* $\mathfrak{R}$ *that are dilutions of other rules.*

- *Define the following operations on pairs of rule schemas:*

$$c_{\mathsf{S}}(\mathfrak{s}_1, \mathfrak{s}_2) := \begin{cases} \dfrac{\Phi_{\mathsf{Y}} \parallel \Phi_{\mathsf{N}}}{\Phi_{\text{⅄}} \parallel \Phi_{\text{И}}} & \textit{if } \mathfrak{s}_1 = \dfrac{\Phi_{\mathsf{Y}}, \varphi \parallel \Phi_{\mathsf{N}}}{\Phi_{\text{⅄}} \parallel \Phi_{\text{И}}} \textit{ and } \mathfrak{s}_2 = \dfrac{\Phi_{\mathsf{Y}} \parallel \Phi_{\mathsf{N}}}{\Phi_{\text{⅄}}, \varphi \parallel \Phi_{\text{И}}} \\ \mathfrak{s}_1 & \textit{otherwise} \end{cases}$$
$$c_{\text{Ƨ}}(\mathfrak{s}_1, \mathfrak{s}_2) := \begin{cases} \dfrac{\Phi_{\mathsf{Y}} \parallel \Phi_{\mathsf{N}}}{\Phi_{\text{⅄}} \parallel \Phi_{\text{И}}} & \textit{if } \mathfrak{s}_1 = \dfrac{\Phi_{\mathsf{Y}} \parallel \Phi_{\mathsf{N}}, \varphi}{\Phi_{\text{⅄}} \parallel \Phi_{\text{И}}} \textit{ and } \mathfrak{s}_2 = \dfrac{\Phi_{\mathsf{Y}} \parallel \Phi_{\mathsf{N}}}{\Phi_{\text{⅄}} \parallel \Phi_{\text{И}}, \varphi} \\ \mathfrak{s}_1 & \textit{otherwise} \end{cases}$$

Then, set

$$C(\mathfrak{R}) := \mathfrak{R} \cup \{R_{c_\alpha(\mathfrak{s}_1, \mathfrak{s}_2)} \mid R_{\mathfrak{s}_1}, R_{\mathfrak{s}_2} \in \mathfrak{R}, \alpha \in \{\mathsf{S}, \text{Ƨ}\}\}$$

and

$$\mathcal{C}(\mathfrak{R}) := \bigcup_{i \in \omega} C^i(\mathfrak{R})$$

The intended streamlining operation $\mathcal{C}^*$*, defined to produce some effect only in systems containing schematic rules (which are the ones we are mostly interested in the present work), consists in applying* $\mathcal{C}$ *to obtain every rule schema resulting from cut for formulas according to* c_{S} *and* $c_{\text{Ƨ}}$*, and then removing the dilutions with* D*:*

$$\mathcal{C}^* := D \circ \mathcal{C}$$

Proposition 91. *Operations O, D and $\mathcal{C}^*$ defined above preserve the* B*-consequence relation induced by the input system, as well as* Θ*-analyticity.*

Proof. Let $\mathfrak{R}$ be a finite and finitary Θ-analytic $\textsc{Set}^2$-$\textsc{Set}^2$ system. Operation O clearly preserves the B-consequence relation induced by $\mathfrak{R}$ as a leaf node can be easily seen to constitute a proof of an instance of overlap, and, by definition, $O(\mathfrak{R}) \subseteq \mathfrak{R}$. Also, any application of an instance of **(O2)** in a Θ-analytic $\mathfrak{R}$-proof can be removed, as this application just produces redundancy by creating a node with the same label as before, and such modification does not affect Θ-analyticity.

In the case of dilution, any application of a dilution of a rule instance r in a Θ-analytic $\mathfrak{R}$-proof may be replaced by an application of r itself, just by removing the nodes resulting from adding formulas not in the succedent of r. In this process, no new formula is introduced, so the proof remains Θ-analytic. Also, by definition, we have $D(\mathfrak{R}) \subseteq \mathfrak{R}$.

Let us now analyze $\mathcal{C}^*$. By the Kleene's fixpoint theorem, $\mathcal{C}(\mathfrak{R})$ is a fixpoint of C. As the rules of inference have finite component sets and c_{S} and $c_{\text{Ƨ}}$ produce rules that have the same or smaller component sets as their inputs, this fixpoint is finite and thus may be generated after a finite number of steps. It remains to prove that $\frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,\mathfrak{R} = \frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,\mathcal{C}(\mathfrak{R})$ and that $\mathcal{C}(\mathfrak{R})$ is Θ-analytic, which follows if we prove that $\frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,\mathfrak{R} = \frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,C^n(\mathfrak{R})$ and that $C^n(\mathfrak{R})$ is Θ-analytic for all $n \in \omega$. By induction on n, the base case $(n = 0)$ is obvious and, by assuming that $\frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,\mathfrak{R} = \frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,C^k(\mathfrak{R})$ and $C^k(\mathfrak{R})$ is Θ-analytic, we just have to show that $\frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,C^k(\mathfrak{R}) \supseteq \frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}\,C^{k+1}(\mathfrak{R})$ and that $C^{k+1}(\mathfrak{R})$ is Θ-analytic. For that effect, suppose that $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}}\big|\frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\,C^{k+1}(\mathfrak{R})$, witnessed by an $\mathfrak{R}$-proof $\mathfrak{t}$. We will show by induction on the structure of $\mathfrak{t}$ that any application of an instance of $\mathfrak{s} := c_{\mathsf{S}}(\mathfrak{s}_1, \mathfrak{s}_2)$, with $R_{\mathfrak{s}_1}, R_{\mathfrak{s}_2} \in C^k(\mathfrak{R})$, can be replaced by applications of rules of inference of $C^k(\mathfrak{R})$ (the case of $c_{\text{Ƨ}}$ will be analogous). In the base case, $\mathfrak{t}$ is a $(\Phi_{\text{⅄}}, \Phi_{\text{И}})$-closed leaf node, so no rule instance was applied and hence it is a proof of the concerned statement in $C^k(\mathfrak{R})$. The case when $\mathfrak{t}$ is a discontinued node can be treated as the case of expanded nodes explained next. Suppose that $\mathfrak{t} = [\Phi_{\mathsf{Y}} \parallel \Phi_{\mathsf{N}} \mid r \mid \mathfrak{t}_1, \ldots, \mathfrak{t}_k]$, where r is an instance of $c_{\mathsf{S}}(\mathfrak{s}_1, \mathfrak{s}_2) = \mathfrak{s}$, $R_{\mathfrak{s}} \notin C^k(\mathfrak{R})$, $\mathfrak{s}_1, \mathfrak{s}_2 \in C^k(\mathfrak{R})$, and each $\mathfrak{t}_i$ is a $(\Phi_{\curlywedge}, \Phi_{\mathsf{И}})$-closed derivation in $C^k(\mathfrak{R})$. We may assume that $r_1 = \frac{\Gamma_{\mathsf{Y}},\varphi \parallel \Gamma_{\mathsf{N}}}{\Gamma_{\curlywedge} \parallel \Gamma_{\mathsf{И}}}$ and $r_2 = \frac{\Gamma_{\mathsf{Y}} \parallel \Gamma_{\mathsf{N}}}{\Gamma_{\curlywedge},\varphi \parallel \Gamma_{\mathsf{И}}}$ are instances of $\mathfrak{s}_1$ and $\mathfrak{s}_2$, respectively, and $r = \frac{\Gamma_{\mathsf{Y}} \parallel \Gamma_{\mathsf{N}}}{\Gamma_{\curlywedge} \parallel \Gamma_{\mathsf{И}}}$. As r was applied to produce $\mathfrak{t}$, we have $\Gamma_{\mathsf{Y}} \subseteq \Phi_{\mathsf{Y}}$ and $\Gamma_{\mathsf{N}} \subseteq \Phi_{\mathsf{N}}$, thus r_2 could have been applied instead. In this case, the only difference would be that a new node would appear in the derivation, having the label $(\Phi_{\mathsf{Y}} \cup \{\varphi\}, \Phi_{\mathsf{N}})$ (the other branches would be closed by the trees $\mathfrak{t}_i$). To the new node, we could apply r_1 and the resulting branches could be closed by the $\mathfrak{R}$-derivations $\mathfrak{t}_i$, but with a little adjustment in the root label, allowed due to **(D2)**. The result is a new tree consisting of a proof in $C^k(\mathfrak{R})$ of the concerned Set^2-Set^2 statement. As $C^k(\mathfrak{R})$ is Θ-analytic by the induction hypothesis, there is a Θ-analytic proof in $C^k(\mathfrak{R})$ of the Set^2-Set^2 statement under concern, which is also a Θ-analytic proof in $C^{k+1}(\mathfrak{R})$, thus $C^{k+1}(\mathfrak{R})$ is also Θ-analytic, and we are done. □

Example 92. *We proceed now to simplify the system produced in Example 82 using the three operations just defined. First of all, by operation O, we remove all rule schemas of group* $(\mathfrak{R}^{\mathfrak{MD}}_{\exists})$ *and all of group* $(\mathfrak{R}^{\mathfrak{MD}}_{\mathsf{D}})$*, with the exception of* $\frac{\parallel}{p \parallel p} E\bot$*. Then, by D, we remove all schemas of group* $(\mathfrak{R}^{\mathfrak{MD}}_{\mathbb{T}})$ *but* $\frac{q \parallel q}{\parallel} E\top$*, as well as all schemas in the group* $(\mathfrak{R}^{\mathfrak{MD}}_{\Sigma})$ *originated from an input tuple containing* $\bot$*, since they constitute dilutions of the latter. That is, the resulting schemas are:*

(x)	$\neg_{\mathbf{A}}(x)$	$A \backslash \neg_{\mathbf{A}}(x)$	Rule schemas
$(\mathbf{f})$	$\{\bot,\mathbf{t}\}$	$\{\mathbf{f}\}$	$\frac{\parallel}{p,\neg p \parallel} \neg^{\mathbf{f}}_{\mathbf{f}}$
$(\mathbf{t})$	$\{\mathbf{f}\}$	$\{\bot, \mathbf{t}\}$	$\frac{p,\neg p \parallel \neg p}{\parallel p} \neg^{\mathbf{t}}_{\bot}$ $\quad$ $\frac{p,\neg p \parallel}{\parallel p,\neg p} \neg^{\mathbf{t}}_{\mathbf{t}}$

(x,y)	$\wedge_{\mathbf{A}}(x,y)$	$A\backslash\wedge_{\mathbf{A}}(x,y)$	Rule schemas
$(\mathbf{f},\mathbf{f})$	$\{\mathbf{f}\}$	$\{\bot,\mathbf{t}\}$	$\dfrac{p\wedge q \parallel p\wedge q}{p,q \parallel}\wedge^{\mathbf{ff}}_{\bot}$ $\dfrac{p\wedge q \parallel}{p,q \parallel p\wedge q}\wedge^{\mathbf{ff}}_{\mathbf{t}}$
$(\mathbf{f},\mathbf{t})$	$\{\mathbf{f}\}$	$\{\bot,\mathbf{t}\}$	$\dfrac{q,p\wedge q \parallel p\wedge q}{p \parallel q}\wedge^{\mathbf{ft}}_{\bot}$ $\dfrac{q,p\wedge q \parallel}{p \parallel q,p\wedge q}\wedge^{\mathbf{ft}}_{\mathbf{t}}$
$(\mathbf{t},\mathbf{f})$	$\{\mathbf{f}\}$	$\{\bot,\mathbf{t}\}$	$\dfrac{p,p\wedge q \parallel p\wedge q}{q \parallel p}\wedge^{\mathbf{tf}}_{\bot}$ $\dfrac{p,p\wedge q \parallel}{q \parallel p,p\wedge q}\wedge^{\mathbf{tf}}_{\mathbf{t}}$
$(\mathbf{t},\mathbf{t})$	$\{\bot,\mathbf{t}\}$	$\{\mathbf{f}\}$	$\dfrac{p,q \parallel}{p\wedge q \parallel p,q}\wedge^{\mathbf{tt}}_{\mathbf{f}}$

(x,y)	$\to_{\mathbf{A}}(x,y)$	$A\backslash\to_{\mathbf{A}}(x,y)$	Rule schemas
$(\mathbf{f},\mathbf{f})$	$\{\bot,\mathbf{t}\}$	$\{\mathbf{f}\}$	$\dfrac{\parallel}{p,q,p\to q \parallel}\to^{\mathbf{ff}}_{\mathbf{f}}$
$(\mathbf{f},\mathbf{t})$	$\{\bot,\mathbf{t}\}$	$\{\mathbf{f}\}$	$\dfrac{q \parallel}{p,p\to q \parallel q}\to^{\mathbf{ft}}_{\mathbf{f}}$
$(\mathbf{t},\mathbf{f})$	$\{\mathbf{f}\}$	$\{\bot,\mathbf{t}\}$	$\dfrac{p,p\to q \parallel p\to q}{q \parallel p}\to^{\mathbf{tf}}_{\bot}$ $\dfrac{p,p\to q \parallel}{q \parallel p,p\to q}\to^{\mathbf{tf}}_{\mathbf{t}}$
$(\mathbf{t},\mathbf{t})$	$\{\bot,\mathbf{t}\}$	$\{\mathbf{f}\}$	$\dfrac{p,q \parallel}{p\to q \parallel p,q}\to^{\mathbf{tt}}_{\mathbf{f}}$

Finally, we may close the resulting collections of schemas under cut and remove the resulting dilutions. In the case of $\neg$*, we may perform a single cut, between schemas* $\neg^{\mathbf{t}}_{\bot}$ *and* $\neg^{\mathbf{t}}_{\mathbf{t}}$*. The resulting schemas are thus:*

$$\frac{\parallel}{p,\neg p \quad \parallel}\neg^{\mathbf{f}}_{\mathbf{f}} \qquad \frac{p,\neg p \quad \parallel}{\parallel \quad p}c\text{ƨ}(\neg^{\mathbf{t}}_{\bot},\neg^{\mathbf{t}}_{\mathbf{t}})$$

In the case of $\wedge$*, we may perform three cuts, resulting in the following schemas:*

$$\frac{p\wedge q \quad \parallel}{p,q \quad \parallel}c\text{ƨ}(\wedge^{\mathbf{ff}}_{\bot},\wedge^{\mathbf{ff}}_{\mathbf{t}}) \qquad \frac{q,p\wedge q \quad \parallel}{p \quad \parallel \quad q}c\text{ƨ}(\wedge^{\mathbf{ft}}_{\bot},\wedge^{\mathbf{ft}}_{\mathbf{t}})$$

$$\frac{p,p\wedge q \quad \parallel}{q \quad \parallel \quad p}c\text{ƨ}(\wedge^{\mathbf{tf}}_{\bot},\wedge^{\mathbf{tf}}_{\mathbf{t}}) \qquad \frac{p,q \quad \parallel}{p\wedge q \quad \parallel \quad p,q}\wedge^{\mathbf{tt}}_{\mathbf{f}}$$

Finally, in the case of $\to$*, we may cut schemas* $\to^{\mathbf{tf}}_{\perp}$ *and* $\to^{\mathbf{tf}}_{\mathbf{t}}$*, and no more. The resulting schemas are thus:*

$$
\frac{\quad \;\|\; \quad}{p, q, p \to q \;\|\;}\to^{\mathbf{ff}}_{\mathbf{f}}
\qquad
\frac{q \;\|\; \quad}{p, p \to q \;\|\; q}\to^{\mathbf{ft}}_{\mathbf{f}}
$$

$$
\frac{p, p \to q \;\|\; \quad}{q \;\|\; p}c_{2}(\to^{\mathbf{tf}}_{\perp}, \to^{\mathbf{tf}}_{\mathbf{t}})
\qquad
\frac{p, q \;\|\; \quad}{p \to q \;\|\; p, q}\to^{\mathbf{tt}}_{\mathbf{f}}
$$

By Proposition 91, then, the resulting system, with 12 rules, is (still) a $\{p\}$*-analytic axiomatization of* $\mathfrak{M}$*.*

We close with other two streamlining procedures. The first one can reduce the number of formulas in a rule schema that has a derivable proper subschema. The second one can reduce the amount of schemas by deleting those schemas which are derivable from the other rules of inference in the system under simplification. In the first case, Θ-analyticity is preserved, while, in the latter, it is preserved in general when the deleted schemas are provable by $\varnothing$-analytic proofs (that is, proofs where only subformulas of the formulas in the schemas appear in the node labels).

Proposition 93. *Let* $\mathfrak{s}_1$ *be a rule schema of* $\mathfrak{R}$ *and a proper dilution of a rule schema* $\mathfrak{s}_2$*. If* $\mathfrak{s}_2$ *is derivable in* $\mathfrak{R}$*, then we may replace* $\mathfrak{s}_2$ *for* $\mathfrak{s}_1$ *in* $\mathfrak{R}$ *preserving the induced* B*-consequence relation. In other words,* ${\div}|{\div}_{\mathfrak{R}} = {\div}|{\div}_{\mathfrak{R}'}$*, where* $\mathfrak{R}' := \mathfrak{R} \backslash \{R_{\mathfrak{s}_1}\} \cup \{R_{\mathfrak{s}_2}\}$*. Moreover, this transformation preserves* Θ*-analyticity.*

Proof. For the first part, note that $\mathfrak{s}_1$ is derivable in $\mathfrak{R}'$ since it is a dilution of $\mathfrak{s}_2$, thus ${\div}|{\div}_{\mathfrak{R}'} = {\div}|{\div}_{\mathfrak{R} \cup \{R_{\mathfrak{s}_2}\}}$ by Proposition 39. Moreover, since $\mathfrak{s}_2$ is derivable in $\mathfrak{R}$, we have ${\div}|{\div}_{\mathfrak{R} \cup \{R_{\mathfrak{s}_2}\}} = {\div}|{\div}_{\mathfrak{R}}$ again by Proposition 39. Thus, ${\div}|{\div}_{\mathfrak{R}} = {\div}|{\div}_{\mathfrak{R}'}$. For the second part, suppose that $\mathfrak{R}$ is Θ-analytic. We want to show that $\mathfrak{R}'$ is also Θ-analytic. Suppose that $\mathfrak{s} := \left(\frac{\Phi_{\text{И}}}{\Phi_{\curlyvee}} \Big| \frac{\Phi_{\curlywedge}}{\Phi_{\mathsf{N}}}\right)$ is provable in $\mathfrak{R}'$. Then it is provable in $\mathfrak{R}$ by a Θ-analytic $\mathfrak{R}$-proof $\mathfrak{t}$. In this proof, we may replace applications of $\mathfrak{s}_1$ by applications of $\mathfrak{s}_2$ (see the proof of

Proposition 91 above with respect to operation D), and this will constitute a Θ-analytic proof of $\mathfrak{s}$ in $\mathfrak{R}'$. $\square$

Proposition 94. *If $\mathfrak{s}_1$ is derivable in $\mathfrak{R}' := \mathfrak{R} \backslash \{R_{\mathfrak{s}_1}\}$, then $\vdots\!|\!\vdots_{\mathfrak{R}} = \vdots\!|\!\vdots_{\mathfrak{R}'}$. Moreover, if there is a $\varnothing$-analytic proof witnessing the latter and $\mathfrak{R}$ is Θ-analytic, then $\mathfrak{R}'$ is Θ-analytic.*

Proof. For the first part, refer to Proposition 39. For the second part, assume that $\mathfrak{R}$ is Θ-analytic and that there is a $\varnothing$-analytic $\mathfrak{R}'$-proof $\mathfrak{t}_{\mathfrak{s}_1}$ of $\mathfrak{s}_1$. Suppose that $\mathfrak{s} := \left(\begin{smallmatrix}\Phi_{\text{И}} & \Phi_{\text{⅄}} \\ \Phi_{\text{Y}} & \Phi_{\text{N}}\end{smallmatrix}\right)$ is provable in $\mathfrak{R}'$. Then it is provable in $\mathfrak{R}$ by a Θ-analytic $\mathfrak{R}$-proof $\mathfrak{t}$. In case in $\mathfrak{t}$ no application of an instance of $\mathfrak{s}_1$ was employed, we are done. Otherwise, use the procedure described in the proof of Proposition 39 to replace applications of instances of $\mathfrak{s}_1$ in $\mathfrak{t}$ by subtrees of $\mathfrak{t}_{\mathfrak{s}_1}$. The fact that $\mathfrak{t}_{\mathfrak{s}_1}$ is $\varnothing$-analytic guarantees that the resulting tree is Θ-analytic, as desired. $\square$

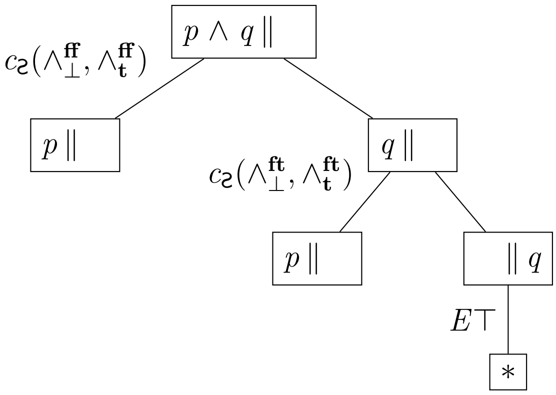
Example 95. *Let us see if the system provided in Example 92 can be further simplified using the two operations defined above. First notice that the subschema $\frac{p, \neg p \ \|}{\|}$ of $c_2(\neg^{\mathbf{t}}_{\bot}, \neg^{\mathbf{t}}_{\mathbf{t}})$ can be easily derived in the system:*

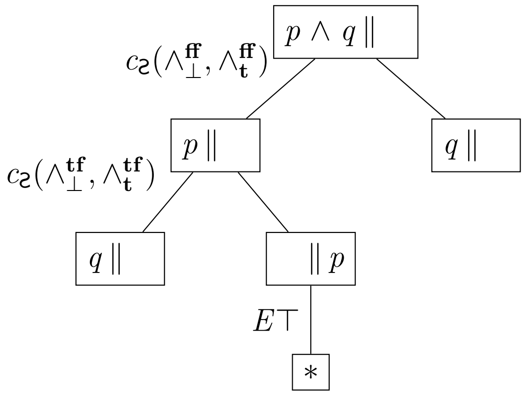
$$
\begin{array}{c}
\boxed{p, \neg p \ \|} \\
c_2(\neg^{\mathbf{t}}_{\bot}, \neg^{\mathbf{t}}_{\mathbf{t}}) \ \big| \\
\boxed{\ \ \| \ p} \\
E\top \ \big| \\
\boxed{*}
\end{array}
$$

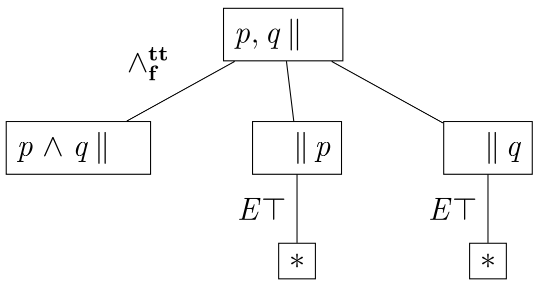
Therefore, by Proposition 93, we may replace it for $c_2(\neg^{\mathbf{t}}_{\bot}, \neg^{\mathbf{t}}_{\mathbf{t}})$. We end up thus with the following couple of rules for $\neg$:

$$
\frac{\quad \|}{p, \neg p \ \ \|}\neg_1 \qquad \frac{p, \neg p \ \ \|}{\quad \|}\neg_2
$$

We try now to do the same for the rules involving $\wedge$. The following trees show, respectively, that the subschemas $\frac{p\wedge q\ \|}{p\ \ \|}$, $\frac{p\wedge q\ \|}{q\ \ \|}$ and $\frac{p,q\ \|}{p\wedge q\ \|}$ are provable in the system:



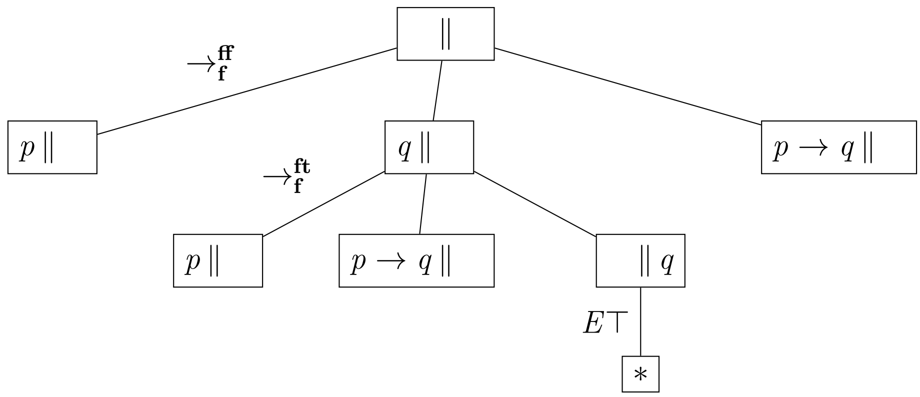


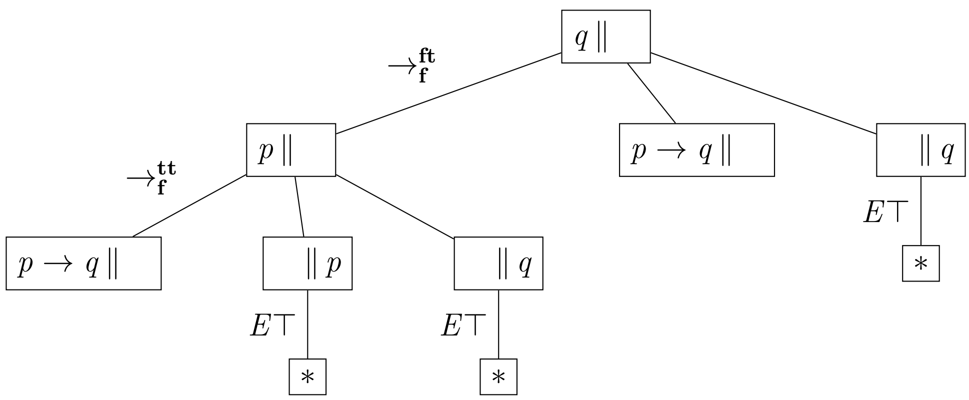


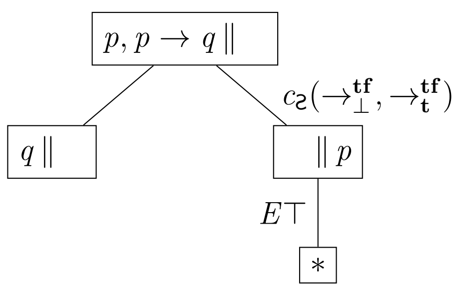
Note that the schema $\frac{p\wedge q\ \|}{p,q\ \ \|}$ is now a dilution of one of the derived subschemas and can, thus, be removed without any harm (equivalently, we could see it as derivable from the other rule schemas and apply Proposition 94). We get, therefore, the following simplified *rules for $\wedge$:*

$$\frac{p \wedge q \quad \|}{p \qquad \|}\wedge_1 \qquad \frac{p \wedge q \quad \|}{q \qquad \|}\wedge_2 \qquad \frac{p,\, q \quad \|}{p \wedge q \quad \|}\wedge_3$$

Now, for the rules involving $\to$, the following derivations show that the subschemas $\frac{\qquad \|}{p, p\to q \ \|}$, $\frac{q \quad \|}{p\to q \ \|}$ and $\frac{p, p\to q \ \|}{q \qquad \|}$ are derivable in the system:







As $\to^{\mathbf{tt}}_{\mathbf{f}}$ *is a dilution of* $\frac{q \quad \|}{p \to q \ \|}$, *we may remove it, and obtain, after all, these three simple rules for* $\to$*:*

$$\frac{\qquad\qquad \|}{p, p \to q \quad \|}\to_1 \qquad \frac{q \qquad \|}{p \to q \quad \|}\to_2 \qquad \frac{p \to q, p \quad \|}{q \qquad \|}\to_3$$

At the end of the day, the obtained system is the same as the one that axiomatizes $\mathfrak{M}_{\{\mathbf{t},\mathbf{f}\}}$*, whose induced* B*-consequence relation has truth-preserving Classical Logic inhabiting the* t*-aspect and falsity-preserving Classical Logic inhabiting the* f*-aspect. In other words, the value* $\perp$ *does not play any role in* $\mathfrak{M}$*. Actually, what we have here is a recipe to produce nd-*B*-matrices of arbitrary sizes for the same* B*-consequence relation: just add new values, and make the interpretations output* $\varnothing$ *whenever these values appear in the input.*

5.4. Extracting a countermodel from a failed proof attempt

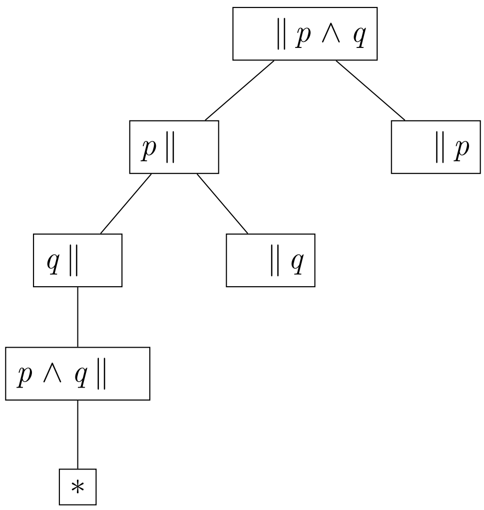
Recall that Proof-Search (Algorithm 4.1) outputs a tree with at least one open branch when the Set^2-Set^2 statement $\mathfrak{s}$ of interest is not provable. From such branch, one may obtain a partition of $\mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$ and, by Proposition 84, define a mapping on $\mathsf{subf}(\mathfrak{s})$ that extends to an $\mathfrak{M}$-valuation. It follows that the discussed algorithm may easily be adapted so as to deliver a countermodel when $\mathfrak{s}$ is unprovable. We formalize and explain this procedure below:

Proposition 96. *Let* $\mathfrak{t}$ *result from a failed proof search attempt of the* B*-statement* $\mathfrak{s} := \left(\begin{array}{c:c} \Phi_{\reflectbox{N}} & \Phi_{\rotatebox[origin=c]{180}{Y}} \\ \hdashline \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}} \end{array}\right)$ *in the system* $\mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$ *using Algorithm 4.1. Then, from* $\mathfrak{t}$*, we may mechanically extract a countermodel for* $\mathfrak{s}$ *in* $\mathfrak{M}$*.*

Proof. By Proposition 75, from $\mathfrak{t}$ we may extract sets $\Psi_{\mathsf{S}}, \Psi_{\reflectbox{S}} \subseteq \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$ such that

$\frac{\mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})\setminus\Psi_{\text{Ƨ}}}{\Psi_{\mathsf{S}}} \ast \frac{\mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})\setminus\Psi_{\mathsf{S}}}{\Psi_{\text{Ƨ}}}\, \mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$. By **(C2)**, then, there are sets $\Psi_{\mathsf{S}} \subseteq \Omega_{\mathsf{S}} \subseteq (\mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})\setminus\Psi_{\mathsf{S}})^{\mathsf{c}}$ and $\Psi_{\text{Ƨ}} \subseteq \Omega_{\text{Ƨ}} \subseteq (\mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})\setminus\Psi_{\text{Ƨ}})^{\mathsf{c}}$ such that $\frac{\Omega^{\mathsf{c}}_{\text{Ƨ}}}{\Omega_{\mathsf{S}}} \ast \frac{\Omega^{\mathsf{c}}_{\mathsf{S}}}{\Omega_{\text{Ƨ}}}\, \mathfrak{R}^{\mathfrak{M}}_{\mathcal{D}}$. By Lemma 84, item 1, we have that, for all $\varphi \in \mathsf{subf}(\mathfrak{s})$, there is an $x_\varphi \in A$ such that (a): $\mathcal{D}^{x_\varphi}_{\alpha}(\varphi) \subseteq \Omega_\beta$ and $\mathcal{D}^{x_\varphi}_{\tilde{\alpha}}(\varphi) \subseteq \Omega^{\mathsf{c}}_{\beta}$, for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \text{Ƨ})\}$. Then, for these values, we also have (b): $\mathcal{D}^{x_\varphi}_{\mathsf{Y}}(\varphi) \subseteq \Psi_{\mathsf{S}}$, $\mathcal{D}^{x_\varphi}_{\text{⅄}}(\varphi) \subseteq \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})\setminus\Psi_{\mathsf{S}}$, $\mathcal{D}^{x_\varphi}_{\mathsf{N}}(\varphi) \subseteq \Psi_{\text{Ƨ}}$ and $\mathcal{D}^{x_\varphi}_{\text{И}}(\varphi) \subseteq \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})\setminus\Psi_{\text{Ƨ}}$. That is, by looking at the sets Ψ_{S} and $\Psi_{\text{Ƨ}}$ and their complements with respect to $\mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s})$, we are able to find values x_φ satisfying (b) and, consequently, satisfying also (a). We can then just build a countermodel as we did in the proof of Theorem 85, that is, by defining a value-assignment $f : \mathsf{subf}(\mathfrak{s}) \to A$ such that $f(\varphi) = x_\varphi$, for each $\varphi \in \mathsf{subf}(\mathfrak{s})$, and considering its extension to an $\mathfrak{M}$-valuation. □

Example 97. *Let us illustrate the countermodel extraction with a simple example, by using the simplified system $\mathfrak{R}$ from Example 95 to extract a countermodel from a failed attempt of proving $\mathfrak{s} := \left(\begin{array}{c|c} p & \\ \hline & p\wedge q \end{array}\right)$ in $\mathfrak{R}$. Note that $\mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s}) = \{p, q, p \wedge q\}$. Here is a failed proof search attempt (notice how the open branch cannot be relevantly expanded by any other instance a schema of $\mathfrak{R}$):*



From the open branch in this tree, we extract the following sets of subformulas: $\Psi_{\mathsf{S}} = \{p\}$ and $\Psi_{\text{Ƨ}} = \{p \wedge q, q\}$. Let us build a value-assignment $f : \mathsf{subf}(\mathfrak{s}) \to \{\mathbf{f}, \bot, \mathbf{t}\}$ to each *subformula according to the Proposition 96:*

- $f(p) = \mathbf{t}$, *since* $\mathcal{D}^{\mathbf{t}}_{\mathsf{Y}}(p) = \{p\} \subseteq \Psi_{\mathsf{S}}$, $\mathcal{D}^{\mathbf{t}}_{\mathsf{\lambda}}(p) = \varnothing \subseteq \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s}) \backslash \Psi_{\mathsf{S}}$, $\mathcal{D}^{\mathbf{t}}_{\mathsf{N}}(p) = \varnothing \subseteq \Psi_{\mathsf{Ƨ}}$ *and* $\mathcal{D}^{\mathbf{t}}_{\mathsf{И}}(p) = \{p\} \subseteq \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s}) \backslash \Psi_{\mathsf{Ƨ}}$.
- $f(q) = \mathbf{f}$, *since* $\mathcal{D}^{\mathbf{f}}_{\mathsf{Y}}(q) = \varnothing \subseteq \Psi_{\mathsf{S}}$, $\mathcal{D}^{\mathbf{f}}_{\mathsf{\lambda}}(q) = \{q\} \subseteq \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s}) \backslash \Psi_{\mathsf{S}}$, $\mathcal{D}^{\mathbf{f}}_{\mathsf{N}}(q) = \varnothing \subseteq \Psi_{\mathsf{Ƨ}}$ *and* $\mathcal{D}^{\mathbf{f}}_{\mathsf{И}}(q) = \varnothing \subseteq \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s}) \backslash \Psi_{\mathsf{Ƨ}}$.
- $f(p \wedge q) = \mathbf{f}$, *since* $\mathcal{D}^{\mathbf{f}}_{\mathsf{Y}}(p \wedge q) = \varnothing \subseteq \Psi_{\mathsf{S}}$, $\mathcal{D}^{\mathbf{f}}_{\mathsf{\lambda}}(p \wedge q) = \{p \wedge q\} \subseteq \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s}) \backslash \Psi_{\mathsf{S}}$, $\mathcal{D}^{\mathbf{f}}_{\mathsf{N}}(p \wedge q) = \varnothing \subseteq \Psi_{\mathsf{Ƨ}}$ *and* $\mathcal{D}^{\mathbf{f}}_{\mathsf{И}}(p \wedge q) = \varnothing \subseteq \mathsf{gsubf}^{\mathcal{D}^{\bowtie}}(\mathfrak{s}) \backslash \Psi_{\mathsf{Ƨ}}$.

Clearly, then, f *extends to an* $\mathfrak{M}$*-valuation constituting a countermodel for* $\mathfrak{s}$ *in* $\mathfrak{M}$.

6. Finite and analytic two-dimensional systems for non-finitely axiomatizable logics

Recall that, in [17], based on the seminal results on axiomatizability via Set-Set H-systems by Shoesmith and Smiley [57], it was proved that any sufficiently expressive non-deterministic logical matrix $\mathbb{M}$ is axiomatizable by a Θ-analytic Set-Set Hilbert-style system, which is finite whenever $\mathbb{M}$ is finite, being Θ a set of separators for the pairs of truth-values of $\mathbb{M}$. We emphasize that it is essential for the above result the adoption of Set-Set H-systems, instead of the more restrict Set-Fmla H-systems. In fact, there are sufficiently expressive nd-matrices that are not finitely axiomatizable by Set-Fmla H-systems, as is witnessed by the three-valued logics presented in [50] (with deterministic two-valued matrices this cannot happen [53]), and by a more recent example in [46], with a simple two-valued non-deterministic matrix. When the nd-matrix at hand is not sufficiently expressive, we may observe the same phenomenon of not having a finite axiomatization in terms of Set-Set H-systems, even if the nd-matrix is finite. The first example (and, to the best of our knowledge, the only one in the current literature) of this fact appeared in [17], which we reproduce here for later reference:

Example 98. *Consider the signature Σ such that $\Sigma_1 := \{g, h\}$ and $\Sigma_k := \varnothing$ for all*

$k \neq 1$. Let $\mathbb{M} := \langle \mathbf{A}, \{\mathbf{t}\}\rangle$ be a Σ-nd-matrix, with $A := \{\mathbf{t}, \mathbf{f}, \bot\}$ and

$$g_{\mathbf{A}}(x) = \begin{cases} \{\mathbf{t}\}, & \textit{if } x = \bot \\ A, & \textit{otherwise} \end{cases} \qquad h_{\mathbf{A}}(x) = \begin{cases} \{\mathbf{f}\}, & \textit{if } x = \mathbf{f} \\ A, & \textit{otherwise} \end{cases}$$

This matrix is not sufficiently expressive because there is no separator for the pair $(\mathbf{f}, \bot)$, and [17] proved that it is not axiomatizable by a finite Set-Set *H-system, even though an infinite* Set-Set *system that captures it has a quite simple description in terms of the following schemas:*

$$\frac{h^i(p)}{p, g(p)}, \ \textit{for all } i \in \omega.$$

In the first section of this chapter, we reveal another example of this same phenomenon, this time of the known logic of formal inconsistency [20] called **mCi**. In the path of proving that this logic is not axiomatizable by a finite Set-Set H-system, we will show that there are infinitely many LFIs between **mbC** and **mCi**, organized in a strictly increasing chain whose limit is **mCi** itself. Then, in the subsequent section, we will show how to obtain a two-dimensional logic inhabited by a (possibly not finitely based) one-dimensional logic of interest. More than that, the obtained logic will be finitely axiomatizable in terms of a Set^2-Set^2 analytic H-system. The only requirements is that the one-dimensional logic of interest must have an associated semantics in terms of a finite non-deterministic logical matrix and that this matrix can be combined with another one through a novel way we will introduce, resulting in a sufficiently expressive nd-B-matrix (recall Section 5.1). This approach will be, in particular, applied here to provide the first finite and analytic axiomatization of **mCi**.

6.1. The logic mCi is not finitely axiomatizable in one dimension

A one-dimensional logic $\rhd$ over Σ is said to be $\neg$-*paraconsistent* when we have $p, \neg p \;\blacktriangleright\; q$, for $p, q \in P$. Moreover, $\rhd$ is $\neg$-*gently explosive* in case there is a collection $\bigcirc(p) \subseteq L_\Sigma(P)$ of unary formulas such that, for some $\varphi \in L_\Sigma(P)$, we have $\bigcirc(\varphi), \varphi \blacktriangleright \varnothing$; $\bigcirc(\varphi), \neg\varphi \blacktriangleright \varnothing$, and, for all $\varphi \in L_\Sigma(P)$, $\bigcirc(\varphi), \varphi, \neg\varphi \;\rhd \varnothing$. We say that $\rhd$ is a *logic of formal inconsistency (**LFI**)* in case it is $\neg$-paraconsistent yet $\neg$-gently explosive. In case $\bigcirc(p) = \{\circ p\}$, for $\circ$ a (primitive or composite) *consistency connective*, the logic is said also to be a ***C**-system*. In what follows, let $\Sigma^{\mathbf{mCi}}$ be the propositional signature such that $\Sigma_1^{\mathbf{mCi}} := \{\neg, \circ\}$, $\Sigma_2^{\mathbf{mCi}} := \{\wedge, \vee, \supset\}$, and $\Sigma_k^{\mathbf{mCi}} := \varnothing$ for all $k \notin \{1, 2\}$.

One of the simplest **C**-systems is the logic **mbC**, which was first presented in terms of a Set-Fmla H-system over $\Sigma^{\mathbf{mCi}}$ obtained by extending any Set-Fmla H-system for positive classical logic ($\mathbf{CPL}^+$) with the following pair of axiom schemas:

(em) $\quad p \vee \neg p$

(bc1) $\quad \circ p \supset (p \supset (\neg p \supset q))$

The logic **mCi**, in turn, is the **C**-system resulting from extending the H-system for **mbC** with the following (infinitely many) axiom schemas [47] (the resulting Set-Fmla H-system is denoted here by $\mathcal{H}_{\mathbf{mCi}}$):

(ci) $\quad \neg\circ p \supset (p \wedge \neg p)$

$(\mathsf{ci})_j$ $\quad \circ\neg^j\circ p$ (for all $0 \leq j < \omega$)

A unary connective © is said to constitute a *classical negation* in a one-dimensional logic $\rhd$ extending $\mathbf{CPL}^+$ in case, for all $\varphi, \psi \in L_\Sigma(P)$, $\varnothing \rhd \varphi \vee ©(\varphi)$ and $\varnothing \rhd \varphi \supset (©(\varphi) \supset \psi)$. One of the main differences with respect to **mbC** is that an inconsistency connective $\bullet$ can be defined using the paraconsistent negation, instead of a classical negation, by setting $\bullet\varphi := \neg\circ\varphi$ [47].

Both logics above were presented in [21] in ways other than H-systems: via tableau systems, via bivaluation semantics and via possible-translation semantics. In addition, despite not being characterizable by a finite deterministic matrix, as shown by Marcos in [47], Arnon Avron in [3] presented a characteristic three-valued nd-matrix for **mbC** and, in [1], a 5-valued non-deterministic logical matrix for **mCi**, witnessing the importance of non-deterministic semantics in the study and applicability of non-classical logics. Such characterizations allow for the extraction of sequent-style systems for these logics by the methodologies developed in [4, 5]. Since **mCi**'s 5-valued nd-matrix will be useful to us in future sections, we describe it again below for ease of reference, now expressing the associated interpretations by truth-tables:

Definition 99. *Let* $\mathcal{V}_5 := \{f, F, I, T, t\}$ *and* $\mathsf{Y}_5 := \{I, T, t\}$. *Define the* $\Sigma^{\mathbf{mCi}}$*-matrix* $\mathbb{M}_{\mathbf{mCi}} := \langle \mathbf{A}_5, \mathsf{Y}_5 \rangle$ *such that* $\mathbf{A}_5 := \langle \mathcal{V}_5, \cdot_{\mathbf{A}_5} \rangle$ *interprets the connectives of* $\Sigma^{\mathbf{mCi}}$ *according to the following (we omit curly braces in the entries of the truth-tables):*

$\wedge_{\mathbf{A}_5}$	f	F	I	T	t
f	f	f	f	f	f
F	f	f	f	f	f
I	f	f	t,I	t,I	t,I
T	f	f	t,I	t,I	t,I
t	f	f	t,I	t,I	t,I

$\vee_{\mathbf{A}_5}$	f	F	I	T	t
f	f	f	t,I	t,I	t,I
F	f	f	t,I	t,I	t,I
I	t,I	t,I	t,I	t,I	t,I
T	t,I	t,I	t,I	t,I	t,I
t	t,I	t,I	t,I	t,I	t,I

$\supset_{\mathbf{A}_5}$	f	F	I	T	t
f	t,I	t,I	t,I	t,I	t,I
F	t,I	t,I	t,I	t,I	t,I
I	f	f	t,I	t,I	t,I
T	f	f	t,I	t,I	t,I
t	f	f	t,I	t,I	t,I

	$\neg_{\mathbf{A}_5}$	$\circ_{\mathbf{A}_5}$
f	t,I	T
F	T	T
I	t,I	F
T	F	T
t	f	T

One might be tempted to apply the axiomatization algorithm of [17] to the finite non-deterministic logical matrix defined above to obtain a finite and analytic SET-SET system for **mCi**. However, it is not obvious at all whether this matrix is sufficiently expressive or not (we will, in fact, prove that it is not). We will show now that **mCi** is actually axiomatizable neither by a finite Set-Fmla H-system (first part), nor by a finite Set-Set H-system (second part); that is, it was not by chance that $\mathcal{H}_{\mathbf{mCi}}$ has been presented with infinitely many rule schemas. Before proceeding to the first part, we note that it is well-known that the collection of all standard consequence relations over a fixed signature constitutes a complete lattice under set-theoretical inclusion. Given a set C of such relations, we will denote by $\bigsqcup C$ its supremum in the latter lattice. We then rely on the following general result for the first part:

Theorem 100 ([64], Theorem 2.2.8, adapted)**.** *Let $\vdash^{\mathsf{t}}$ be a standard (that is, finitary and substitution-invariant)* Set-Fmla *consequence relation. Then $\vdash^{\mathsf{t}}$ is axiomatizable by a finite* Set-Fmla *H-system if, and only if, there is no strictly increasing sequence $\vdash^{\mathsf{t}}_{0}, \vdash^{\mathsf{t}}_{1}, \ldots, \vdash^{\mathsf{t}}_{n}, \ldots$ of standard* Set-Fmla *consequence relations such that $\vdash^{\mathsf{t}} = \bigsqcup_{i \in \omega} \vdash^{\mathsf{t}}_{i}$.*

In order to apply the above theorem, we present a family of finite Set-Fmla H-systems that, in the sequel, will be used to provide an increasing sequence of standard Set-Fmla consequence relations whose supremum is precisely **mCi**. Then, we will show that this sequence is strictly increasing, by employing the matrix methodology traditionally used for showing the independence of axioms in a proof system.

Definition 101. *For each $k \in \omega$, let $\mathcal{H}^{k}_{\mathbf{mCi}}$ be a* Set-Fmla *H-system for positive classical logic together with the schemas* (em)*,* (bc1)*,* (ci) *and* $(\mathsf{ci})_j$*, for all $0 \leq j \leq k$.*

Since $\mathcal{H}^{k}_{\mathbf{mCi}}$ may be obtained from $\mathcal{H}_{\mathbf{mCi}}$ by deleting some (infinitely many) axioms, it is immediate that:

Proposition 102. *For every $k \in \omega$, $\vdash^{\mathsf{t}}_{\mathcal{H}^{k}_{\mathbf{mCi}}} \subseteq \vdash^{\mathsf{t}}_{\mathbf{mCi}}$.*

The way we define the promised increasing sequence of consequence relations in the next result is by taking the systems $\mathcal{H}^{k}_{\mathbf{mCi}}$ with odd superscripts, namely, we will be working with the sequence $\vdash^{\mathsf{t}}_{\mathcal{H}^{1}_{\mathbf{mCi}}}, \vdash^{\mathsf{t}}_{\mathcal{H}^{3}_{\mathbf{mCi}}}, \vdash^{\mathsf{t}}_{\mathcal{H}^{5}_{\mathbf{mCi}}}, \ldots$ Excluding the cases where k is even will facilitate, in particular, the proof of Lemma 106, where we show that the proposed sequence of logics is *strictly* increasing.

Lemma 103. *For each* $1 \leq k < \omega$, *let* $\vdash^{\mathsf{t}}_{k} := \vdash^{\mathsf{t}}_{\mathcal{H}^{2k-1}_{\mathbf{mCi}}}$. *Then* $\vdash^{\mathsf{t}}_{1} \subseteq \vdash^{\mathsf{t}}_{2} \subseteq \ldots$, *and*

$$\vdash^{\mathsf{t}}_{\mathbf{mCi}} = \bigsqcup_{1 \leq k < \omega} \vdash^{\mathsf{t}}_{k}.$$

Proof. By Definition 101, every rule schema in $\mathcal{H}^{2k-1}_{\mathbf{mCi}}$ is also in $\mathcal{H}^{2(k+1)-1}_{\mathbf{mCi}}$, thus, for every $1 \leq k < \omega$, we have $\vdash^{\mathsf{t}}_{k} \subseteq \vdash^{\mathsf{t}}_{k+1}$. Let $\vdash^{\mathsf{t}}_{\omega} := \bigsqcup_{1 \leq k < \omega} \vdash^{\mathsf{t}}_{k}$. From right to left, if $\Phi \vdash^{\mathsf{t}}_{\omega} \psi$, then, for every SET-FMLA consequence relation $\vdash^{\mathsf{t}}_{*}$ over $\Sigma^{\mathbf{mCi}}$ such that $\vdash^{\mathsf{t}}_{*} \supseteq \vdash^{\mathsf{t}}_{k}$ for all $k \in \omega$, we have $\Phi \vdash^{\mathsf{t}}_{*} \psi$. By Proposition 102, then, we have $\Phi \vdash^{\mathsf{t}}_{\mathbf{mCi}} \psi$, in particular. From left to right, suppose that $\Phi \vdash^{\mathsf{t}}_{\mathbf{mCi}} \psi$ and consider a derivation bearing witness to this fact. Let $m \in \omega$ be such that only instances of the rule schemas $(\mathsf{ci})_j$, for $0 \leq j \leq m$, and possibly instances of the other rule schemas not of the form of $(\mathsf{ci})_j$ are applied in that derivation. Let $\vdash^{\mathsf{t}}_{**}$ be a SET-FMLA consequence relation over $\Sigma^{\mathbf{mCi}}$ such that $\vdash^{\mathsf{t}}_{**} \supseteq \vdash^{\mathsf{t}}_{k}$ for all $1 \leq k < \omega$. Then, in particular, $\vdash^{\mathsf{t}}_{**} \supseteq \vdash^{\mathsf{t}}_{m} = \vdash^{\mathsf{t}}_{\mathcal{H}^{2m-1}_{\mathbf{mCi}}}$. Since all schemas $(\mathsf{ci})_j$, for $0 \leq j \leq m$, are in $\mathcal{H}^{2m-1}_{\mathbf{mCi}}$, we have $\Phi \vdash^{\mathsf{t}}_{\mathcal{H}^{2m-1}_{\mathbf{mCi}}} \psi$ and then $\Phi \vdash^{\mathsf{t}}_{**} \psi$. As $\vdash^{\mathsf{t}}_{**}$ was an arbitrary upper bound, the result applies, in particular, to the least upper bound $\vdash^{\mathsf{t}}_{\omega}$, and we are done. □

Finally, we prove that the sequence outlined in the paragraph before Lemma 103 is strictly increasing. In order to achieve this, we define, for each $1 \leq k < \omega$, a $\Sigma^{\mathbf{mCi}}$-matrix $\mathbb{M}_k$ and prove that $\mathcal{H}^{2k-1}_{\mathbf{mCi}}$ is sound with respect to such matrix. Then, in the second part of the proof (the "independence part"), we show that, for each $1 \leq k < \omega$, $\mathbb{M}_k$ fails to validate the rule schema $(\mathsf{ci})_j$, for $j = 2k$, which is present in $\mathcal{H}^{2(k+1)-1}_{\mathbf{mCi}}$. In this way, by the contrapositive of the soundness result proved in the first part, we will have $(\mathsf{ci})_j$ provable in $\mathcal{H}^{2(k+1)-1}_{\mathbf{mCi}}$ while unprovable in $\mathcal{H}^{2k-1}_{\mathbf{mCi}}$. In what follows, for any $k \in \omega$, we use k^* to refer to the successor of k.

Definition 104. *Let* $1 \leq k < \omega$. *Define the* $2k^*$*-valued* $\Sigma^{\mathbf{mCi}}$*-matrix* $\mathbb{M}_k := \langle \mathbf{A}_k, D_k \rangle$

such that $D_k := \{k^*+1, \ldots, 2k^*\}$ *and* $\mathbf{A}_k := \langle\{1, \ldots, 2k^*\}, \cdot_{\mathbf{A}_k}\rangle$*, the interpretation of* $\Sigma^{\mathbf{mCi}}$ *in* $\mathbf{A}_k$ *given by the following operations:*

$$x \vee_{\mathbf{A}_k} y := \begin{cases} 1 & \textit{if } x, y \in \overline{D_k} \\ k^*+1 & \textit{otherwise} \end{cases} \qquad x \wedge_{\mathbf{A}_k} y := \begin{cases} k^*+1 & \textit{if } x, y \in D_k \\ 1 & \textit{otherwise} \end{cases}$$

$$x \supset_{\mathbf{A}_k} y := \begin{cases} 1 & \textit{if } x \in D_k \textit{ and } y \notin \overline{D_k} \\ k^*+1 & \textit{otherwise} \end{cases}$$

$$\circ_{\mathbf{A}_k} x := \begin{cases} 1 & \textit{if } x = 2k^* \\ k^*+1 & \textit{otherwise} \end{cases} \quad \neg_{\mathbf{A}_k} x := \begin{cases} k^*+1 & \textit{if } x \in \{1, 2k^*\} \\ x + k^* & \textit{if } 2 \leq x \leq k^* \\ x - (k^*-1) & \textit{if } k^*+1 \leq x \leq 2k^*-1 \end{cases}$$

We illustrate in Figure 6.1 the general form of those operations, in an attempt to make explicit the similarity between the interpretations of $\wedge$, $\vee$ *and* $\supset$ *and the usual two-valued interpretations for such connectives in positive classical logic, as well as to facilitate the understanding of the interpretations of* $\neg$ *and* $\circ$.

$\vee_{\mathbf{A}_k}$	1	$\dots$	k^*	k^*+1	$\dots$	$2k^*$
1	1	$\dots$	1			
$\vdots$	$\vdots$	$\ddots$	$\vdots$			
k^*	1	$\dots$	1			
k^*+1						
$\vdots$					k^*+1	
$2k^*$						

$\wedge_{\mathbf{A}_k}$	1	$\dots$	k^*	k^*+1	$\dots$	$2k^*$
1						
$\vdots$		k^*+1				
k^*						
k^*+1				1	$\dots$	1
$\vdots$				$\vdots$	$\ddots$	$\vdots$
$2k^*$				1	$\dots$	1

$\supset_{\mathbf{A}_k}$	1	$\dots$	k^*	k^*+1	$\dots$	$2k^*$
1						
$\vdots$					k^*+1	
k^*						
k^*+1	1	$\dots$	1			
$\vdots$	$\vdots$	$\ddots$	$\vdots$			
$2k^*$	1	$\dots$	1			

	$\circ_{\mathbf{A}_k}$
1	k^*+1
2	k^*+1
$\vdots$	$\vdots$
k^*	k^*+1
k^*+1	k^*+1
$\vdots$	$\vdots$
$2k^*-1$	k^*+1
$2k^*$	1

	$\neg_{\mathbf{A}_k}$
1	k^*+1
2	k^*+2
$\vdots$	$\vdots$
k^*	$2k^*$
k^*+1	2
$\vdots$	$\vdots$
$2k^*-1$	k^*
$2k^*$	k^*+1

Figure 6.1.: Illustration of the interpretations of the connectives in $\Sigma^{\mathbf{mCi}}$ provided by the matrix $\mathbb{M}_k$.

Before continuing, we prove a result about this construction, which will be used in the remainder of the current line of argumentation. In what follows, when there is no risk of confusion, we omit the subscript '$\mathbf{A}_k$' from the interpretations to simplify the notation.

Lemma 105. *For all $k \geq 1$ and $1 \leq m \leq 2k$,*

$$\neg^m_{\mathbf{A}_k}(k^* + 1) = \begin{cases} (k^* + 1) + \frac{m}{2}, & \textit{if } m \textit{ is even} \\ 1 + \frac{m+1}{2}, & \textit{otherwise} \end{cases}$$

Proof. Let $k \geq 1$. We prove the lemma by strong induction on $1 \leq m \leq 2k$. For $m = 1$, we have $\neg(k^* + 1) = (k^* + 1) - (k^* - 1) = 2 = 1 + \frac{1+1}{2}$. Assume now that (IH): the present lemma holds for all $m' < m$, for a given $m > 1$.

- Suppose that $m = 2s$, with $1 \leq s \leq k$. By (IH), we have that $\neg^{2s}(k^* + 1) = \neg(\neg^{2s-1}(k^* + 1)) = \neg(1 + \frac{(2s-1)+1}{2}) = \neg(1 + s)$. By the interpretation of $\neg$, as $2 \leq 1 + s \leq k^*$, we have $\neg(1 + s) = 1 + s + k^* = (k^* + 1) + \frac{m}{2}$.
- Suppose that $m = 2s + 1$, with $1 \leq s \leq k - 1$. By (IH), we have $\neg^{2s+1}(k^* + 1) = \neg(\neg^{2s}(k^* + 1)) = \neg(k^* + 1 + \frac{2s}{2}) = \neg(k^* + 1 + s)$. As $k^* + 2 \leq k^* + 1 + s \leq k^* + k$, the interpretation of $\neg$ gives us that $\neg(k^* + 1 + s) = (k^* + 1 + s) - (k^* - 1) = s + 2 = \frac{m-1}{2} + 2 = (\frac{m-1}{2} + 1) + 1 = 1 + \frac{m+1}{2}$.

$\square$

Lemma 106. *For all $1 \leq k < \omega$, we have $\vdash^{\mathsf{t}}_{\mathcal{H}^{2k^*-1}_{\mathbf{mCi}}} \circ\neg^{2k}\circ p$ but $\nvdash^{\mathsf{t}}_{\mathcal{H}^{2k-1}_{\mathbf{mCi}}} \circ\neg^{2k}\circ p$.*

Proof. Let $1 \leq k < \omega$. We begin by showing that $\mathcal{H}^{2k-1}_{\mathbf{mCi}}$ is sound for $\mathbb{M}_k$. The rule schemas from positive classical logic are sound with respect to $\mathbb{M}_k$, since the mapping h given by $h(x) = \mathbf{F}$ if $x \in \{1, \ldots, k^*\}$ and $h(x) = \mathbf{T}$ otherwise is a strong homomorphism from the positive fragment of $\mathbb{M}_k$ onto $\mathbb{B}$, the usual two-valued matrix that determines positive classical logic. Below we show soundness of the remaining rules (all of which are axiom schemas), which involve connectives $\neg$ and $\circ$. The lemma just proved will be employed in the case of (ci)$_j$.

(em) Suppose that $v(\varphi \vee \neg\varphi) \in \overline{D_k}$, then $v(\varphi) \in \overline{D_k}$ and $v(\neg\varphi) \in \overline{D_k}$. From the latter, we have $k^* + 1 \leq v(\varphi) \leq 2k^* - 1$, but then $v(\varphi) \in D_k$, a contradiction.

(bc1) Suppose that $v(\circ\varphi\supset(\varphi\supset(\neg\varphi\supset\psi))) \in \overline{D_k}$. Then (a): $v(\circ\varphi) \in D_k$ and $v(\varphi\supset(\neg\varphi\supset\psi)) \in \overline{D_k}$. From the latter, reasoning in the same way, we have (b): $v(\varphi) \in D_k$, (c): $v(\neg\varphi) \in D_k$ and $v(\psi) \in \overline{D_k}$. From (b), (c) and the interpretation of $\neg$, we have that $v(\varphi) = 2k^*$, but then $v(\circ\varphi) = 1 \in \overline{D_k}$, contradicting (a).

(ci) Suppose that $v(\neg\circ\varphi\supset(\varphi \wedge \neg\varphi)) \in \overline{D_k}$. Then (a): $v(\neg\circ\varphi) \in D_k$ and (b): $v(\varphi \wedge \neg\varphi) \in \overline{D_k}$. From (a), we have (c): $1 \leq v(\circ\varphi) \leq k^*$ or $v(\circ\varphi) = 2k^*$. From (b), we have that either (b1): $v(\varphi) \in \overline{D_k}$ or (b2): $v(\neg\varphi) \in \overline{D_k}$. By cases:

- if (b1), then $v(\circ\varphi) = k^* + 1$, contradicting (c).
- if (b2), then $k^* + 1 \leq v(\varphi) \leq 2k^* - 1$ by the interpretation of $\neg$, but then $v(\circ\varphi) = k^* + 1$ by the interpretation of $\circ$, contradicting (c).

(ci)$_j$ For $j = 0$, suppose that $v(\circ\circ\varphi) \in \overline{D_k}$. Then, $v(\circ\varphi) = 2k^*$, which is impossible from the interpretation of $\circ$. Let $1 \leq j \leq 2k - 1$. Suppose that $v(\circ\neg^j\circ\varphi) \in \overline{D_k}$. Then, by the interpretation of $\circ$, we have (a): $v(\neg^j\circ\varphi) = 2k^*$. By cases on the possible values of $v(\circ\varphi)$:

- if $v(\circ\varphi) = k^* + 1$: by Lemma 105, if j is even, we have $\neg^j(k^* + 1) = (k^* + 1) + \frac{j}{2} = (k^* + 1) + s = k + 2 + s \leq k + 2 + k - 1 = 2k + 1 < 2k^*$, with $1 \leq s \leq k - 1$. If j is odd, then $\neg^j(k^* + 1) = 1 + \frac{j+1}{2} = 1 + \frac{2s-1+1}{2} = 1 + s \leq 1 + k < 2k^*$, with $1 \leq s \leq k$. Both cases contradict (a).
- if $v(\circ\varphi) = 1$: we may apply the same reasoning of the previous item, since $v(\neg^j\circ\varphi) = \neg^{j-1}v(\neg\circ\varphi) = \neg^{j-1}(k^* + 1)$.

For the second part of the proof, take a $\mathbb{M}_k$-valuation v such that $v(p) = 1$. Then

$v(\circ p) = k^* + 1$ and, since $2k$ is even, by Lemma 105, we have $\neg^{2k}(k^* + 1) = (k^* + 1) + \frac{2k}{2} = k^* + 1 + k = 2k + 2 = 2k^*$. Thus $v(\neg^{2k}\circ p) = 2k^*$ and, by the interpretation of $\circ$, we have $v(\circ\neg^{2k}\circ p) = 1 \in \overline{D_k}$, and we are done. $\square$

Finally, Theorem 100, Lemma 103 and Lemma 106 give us the main result:

Theorem 107. **mCi** *is not axiomatizable by a finite* Set-Fmla *H-system.*

For the second part —namely, that no finite Set-Set H-system axiomatizes **mCi**—, we explore the following result:

Theorem 108 ([57], Theorem 5.37, adapted)**.** *Let* $\rhd^{\mathsf{t}}$ *be a one-dimensional consequence relation over a propositional signature containing the binary connective* $\vee$*. Suppose that the* Set-Fmla *companion of* $\rhd^{\mathsf{t}}$*, denoted by* $\vdash_{\rhd^{\mathsf{t}}}$*, satisfies the following property:*

$$\Phi, \varphi \vee \psi \vdash_{\rhd^{\mathsf{t}}} \gamma \textit{ if, and only if, } \Phi, \varphi \vdash_{\rhd^{\mathsf{t}}} \gamma \textit{ and } \Phi, \psi \vdash_{\rhd^{\mathsf{t}}} \gamma \tag{Disj}$$

If a Set-Set *H-system* R *axiomatizes* $\rhd^{\mathsf{t}}$*, then* R *may be converted into a* Set-Fmla *H-system for* $\vdash_{\rhd^{\mathsf{t}}}$ *that is finite whenever* R *is finite.*

It turns out that:

Lemma 109. **mCi** *satisfies* (Disj).

Proof. The non-deterministic semantics of **mCi** gives us that, for all $\varphi, \psi \in L_{\Sigma^{\mathbf{mCi}}}(P)$, $\varphi \rhd^{\mathsf{t}}_{\mathbb{M}_{\mathbf{mCi}}} \varphi \vee \psi$; $\psi \rhd^{\mathsf{t}}_{\mathbb{M}_{\mathbf{mCi}}} \varphi \vee \psi$, and $\varphi \vee \psi \rhd^{\mathsf{t}}_{\mathbb{M}_{\mathbf{mCi}}} \varphi, \psi$, and such facts easily imply (Disj) by **(T)**. $\square$

Theorem 110. **mCi** *is not axiomatizable by a finite* Set-Set *H-system.*

Proof. If R were a finite Set-Set H-system for **mCi**, then, by Lemma 109 and Theorem 108, it could be turned into a finite Set-Fmla H-system for this very logic, which contradicts Theorem 107. $\square$

Finding a finite one-dimensional H-system for **mCi** (analytic or not) over the same language, then, proved to be impossible. The previous result also tells us that there is no sufficient expressive non-deterministic matrix that characterizes **mCi** (for otherwise the recipe in [17] would deliver a finite analytic SET-SET H-system for it), and we may conclude, in particular, that:

Corollary 111. *The nd-matrix* $\mathbb{M}_{\mathbf{mCi}}$ *is not sufficiently expressive.*

The pairs of truth-values of $\mathbb{M}_{\mathbf{mCi}}$ that seem not to be separable (at least one of these pairs must not be, in view of the above corollary) are (t, T) and (f, F). The insufficiency of expressive power to take these specific pairs of values apart, however, would not occur if we had considered instead the matrix defined below:

Definition 112. *Let* $\mathbb{M}^{\mathsf{n}}_{\mathbf{mCi}} := \langle \mathbf{A}_5, \mathsf{N}_5 \rangle$*, where* $\mathsf{N}_5 := \{f, I, T\}$*.*

Note that $t \notin \mathsf{N}_5$, while $T \in \mathsf{N}_5$, and that $f \in \mathsf{N}_5$, while $F \notin \mathsf{N}_5$. Therefore, the single propositional variable p separates in $\mathbb{M}^{\mathsf{n}}_{\mathbf{mCi}}$ the pairs (t, T) and (f, F). On the other hand, it is not clear now whether the pairs (t, F) and (f, T) are separable in this new matrix. Nonetheless, we will see, in the next section, how we can take advantage of the semantics of non-deterministic B-matrices in order to combine the expressiveness of $\mathbb{M}_{\mathbf{mCi}}$ and $\mathbb{M}^{\mathsf{n}}_{\mathbf{mCi}}$ in a very simple and intuitive manner, preserving the language and the algebra shared by these matrices. We identify two important aspects of this combination: first, the logics determined by the original matrices can be fully recovered from the combined logic; and, second, since the notions of H-systems and sufficient expressiveness, as well as the axiomatization algorithm of [17], were generalized in Chapter 5, the resulting two-dimensional logic may be algorithmically axiomatized by an *analytic* two-dimensional H-system that is *finite* if the combining matrices are finite, provided the criterion of sufficient expressiveness is satisfied after the combination. This will be particularly the case when we combine $\mathbb{M}_{\mathbf{mCi}}$ and $\mathbb{M}^{\mathsf{n}}_{\mathbf{mCi}}$. Consequently, this novel way of combining logics provides a quite general approach for producing finite and analytic axiomatizations for logics determined by non-deterministic logical matrices that may be not finitely axiomatizable in one dimension; in particular, the logics from Example 98 and **mCi**.

6.2. Combining two logical matrices into a nd-B-matrix

In Section 5.2, the axiomatization algorithm of [17] was generalized to nd-B-matrices and $\textsc{Set}^2$-$\textsc{Set}^2$ H-systems, guaranteeing that every sufficiently expressive nd-B-matrix $\mathfrak{M}$ is axiomatizable by a Θ-analytic $\textsc{Set}^2$-$\textsc{Set}^2$ H-system, which is finite whenever $\mathfrak{M}$ is finite, where Θ is a set of separators for the pairs of truth-values of $\mathfrak{M}$. Note that, in view of the generalization of the property of sufficient expressiveness provided in Section 5.1, a unary formula is characterized as a separator whenever it separates a pair of truth-values according to *at least one* of the distinguished sets of values. This means that having two of such sets may allow us to separate more pairs of truth-values than having a single set, that is, the nd-B-matrices are, in this sense, potentially more expressive than the (one-dimensional) logical matrices.

Example 113. *Let* $\mathbf{A}$ *be the* Σ*-nd-algebra from Example 98, and consider the nd-*B*-matrix* $\mathfrak{M} := \langle \mathbf{A}, \{\mathbf{t}\}, \{\mathbf{f}\}\rangle$*. As we know, in this matrix the pair* $(\mathbf{f}, \bot)$ *is not separable if we consider only the set of designated values* $\{\mathbf{t}\}$*. However, as we have now the set* $\{\mathbf{f}\}$ *of antidesignated truth-values, the separation becomes evident: the propositional variable* p *is a separator for this pair now, since* $\mathbf{f} \in \{\mathbf{f}\}$ *and* $\bot \notin \{\mathbf{f}\}$*. The recipe in Definition 81 together with the simplifications described in Section 5.3, then, produce the following* $\textsc{Set}^2$-$\textsc{Set}^2$ *axiomatization for* $\mathfrak{M}$*, with only three very simple schematic rules of inference:*

$$\frac{p \;\|\; p}{\|} \qquad \frac{\|}{f(p), p \;\|\; p} \qquad \frac{\|\; p}{\|\; t(p)}$$

By construction, the one-dimensional logic determined by the nd-matrix of Example 98 inhabits the t*-aspect of* $\div\!|\!\div_{\mathfrak{M}}$*, thus it can be seen as being axiomatized by this* finite *and* analytic *two-dimensional system (contrast with the* infinite SET-SET *axiomatization known for this logic provided in that same example).*

We constructed above a Σ-nd-B-matrix from two Σ-nd-matrices in such a way that the one-dimensional logics determined by latter are fully recoverable from the former. We formalize this construction below:

Definition 114. *Let* $\mathbb{M} := \langle \mathbf{A}, D \rangle$ *and* $\mathbb{M}' := \langle \mathbf{A}, D' \rangle$ *be* Σ*-nd-matrices. The* B-product *between* $\mathbb{M}$ *and* $\mathbb{M}'$ *is the* Σ*-nd-*B*-matrix* $\mathbb{M} \odot \mathbb{M}' := \langle \mathbf{A}, D, D' \rangle$*.*

Note that $\Phi \rhd^{\mathsf{t}}_{\mathbb{M}} \Psi$ iff $\frac{\ }{\Phi}|\frac{\Psi}{\ }_{\mathbb{M}\odot\mathbb{M}'}$ iff $\Phi \rhd^{\mathbb{M}\odot\mathbb{M}'}_{\mathsf{t}} \Psi$, and $\Phi \rhd^{\mathsf{t}}_{\mathbb{M}'} \Psi$ iff $\frac{\Psi}{\ }|\frac{\ }{\Phi}_{\mathbb{M}\odot\mathbb{M}'}$ iff $\Phi \rhd^{\mathbb{M}\odot\mathbb{M}'}_{\mathsf{f}} \Psi$. Therefore, $\rhd^{\mathsf{t}}_{\mathbb{M}}$ and $\rhd^{\mathsf{t}}_{\mathbb{M}'}$ are easily recoverable from $\div\!|\!\div_{\mathbb{M}\odot\mathbb{M}'}$, since they inhabit, respectively, the t-aspect and the f-aspect of the latter. One of the applications of this novel way of putting two distinct logics together was illustrated in that same Example 113 to produce a two-dimensional analytic and finite axiomatization for a one-dimensional logic characterized by a Σ-nd-matrix. As we have shown, the one-dimensional logic does not need to be finitely axiomatizable by a SET-SET H-system. We present this application of B-products with more generality below:

Proposition 115. *Let* $\mathbb{M} := \langle \mathbf{A}, D \rangle$ *be a* Σ*-nd-matrix and suppose that the pairs* $(x_1, y_1), \ldots, (x_n, y_n), \ldots \in A \times A$ *of distinct truth-values are not separable in* $\mathbb{M}$*. If, for some* $\mathbb{M}' := \langle \mathbf{A}, D' \rangle$*, these pairs are separable in* $\mathbb{M}'$*, then* $\mathbb{M} \odot \mathbb{M}'$ *is sufficiently expressive (thus, axiomatizable by an analytic* SET^2-SET^2 *H-system, that is finite whenever* $\mathbf{A}$ *is finite).*

Proof. Let $(z, w) \in A \times A$. In case $(z, w) \neq (x_i, y_i)$ for all i, there is a separator S for (z, w)

in $\mathbb{M}$, that is, $\mathrm{S}_{\mathbf{A}}(z)\#_{D}\mathrm{S}_{\mathbf{A}}(w)$. Otherwise, if (x_i, y_i) is separable in $\mathbb{M}'$ for all i, then, in particular, (z, w) is also separable in $\mathbb{M}'$, say by a separator S', that is, $\mathrm{S}'_{\mathbf{A}}(z)\#_{D'}\mathrm{S}'_{\mathbf{A}}(w)$. Therefore, every pair of truth-values of $\mathbf{A}$ is separable in $\mathfrak{M}$, thus the latter is sufficiently expressive. By the procedure in Section 5.2, $\mathfrak{M}$ is axiomatizable by an analytic $\textsc{Set}^2$-$\textsc{Set}^2$ H-system that is finite if $\mathbf{A}$ is finite. $\square$

Let us, in the next subsection, return to the case of **mCi** and see how the latter result can be applied to finitely axiomatize this logic in two dimensions.

6.3. A finite and analytic two-dimensional system for mCi

In the spirit of Proposition 115, we define below a $\Sigma^{\mathbf{mCi}}$-nd-B-matrix by combining the matrices $\mathbb{M}_{\mathbf{mCi}} := \langle \mathbf{A}_5, \mathsf{Y}_5 \rangle$ and $\mathbb{M}^{\mathsf{n}}_{\mathbf{mCi}} := \langle \mathbf{A}_5, \mathsf{N}_5 \rangle$ introduced respectively in Definition 99 and Definition 112:

Definition 116. *Let* $\mathfrak{M}_{\mathbf{mCi}} := \mathbb{M}_{\mathbf{mCi}} \odot \mathbb{M}^{\mathsf{n}}_{\mathbf{mCi}} = \langle \mathbf{A}_5, \mathsf{Y}_5, \mathsf{N}_5 \rangle$*, with* $\mathsf{Y}_5 := \{I, T, t\}$ *and* $\mathsf{N}_5 := \{f, I, T\}$.

When we consider now both sets Y_5 and N_5 of designated and antidesignated truth-values, the separation of all truth-values of $\mathbf{A}_5$ becomes possible, that is, $\mathfrak{M}_{\mathbf{mCi}}$ is sufficiently expressive, as guaranteed by Proposition 115. Furthermore, notice that we have two alternatives for separating the pairs (I, t) and (I, T): either using the formula $\neg p$ or the formula $\circ p$. With this finite sufficiently expressive nd-B-matrix in hand, producing a *finite* $\{p, \circ p\}$-analytic two-dimensional H-system for it is immediate by the results in Section 5.2. Therefore,

Theorem 117. **mCi** *is axiomatizable by a finite and analytic two-dimensional H-system.*

The axiomatization recipe in Definition 81 delivers a $\textsc{Set}^2$-$\textsc{Set}^2$ H-system with about 300 rule schemas. When we simplify it using the streamlining procedures indicated in Section 5.3, we obtain a much more succinct and insightful presentation, with 28 rule schemas, which we call $\mathfrak{R}_{\mathbf{mCi}}$ and present in full below:

$\mathfrak{R}_{\mathbf{mCi}}$

$$\frac{q \;\parallel}{p\supset q \;\parallel}\supset_1^{\mathbf{mCi}} \qquad \frac{\parallel}{p,\, p\supset q \;\parallel}\supset_2^{\mathbf{mCi}} \qquad \frac{p\supset q,\, p \;\parallel}{q \;\parallel}\supset_3^{\mathbf{mCi}}$$

$$\frac{p \;\parallel}{q \;\parallel\; p\supset q}\supset_4^{\mathbf{mCi}} \qquad \frac{p\supset q,\, \circ(p\supset q) \;\parallel\; p\supset q}{\parallel}\supset_5^{\mathbf{mCi}}$$

$$\frac{p,\, q \;\parallel}{p \wedge q \;\parallel}\wedge_1^{\mathbf{mCi}} \qquad \frac{p \wedge q \;\parallel}{p \;\parallel}\wedge_2^{\mathbf{mCi}} \qquad \frac{p \wedge q \;\parallel}{q \;\parallel}\wedge_3^{\mathbf{mCi}}$$

$$\frac{\parallel}{p \wedge q \;\parallel\; p \wedge q}\wedge_4^{\mathbf{mCi}} \qquad \frac{p \wedge q,\, \circ(p \wedge q) \;\parallel\; p \wedge q}{\parallel}\wedge_5^{\mathbf{mCi}}$$

$$\frac{p \;\parallel}{p \vee q \;\parallel}\vee_1^{\mathbf{mCi}} \qquad \frac{q \;\parallel}{p \vee q \;\parallel}\vee_2^{\mathbf{mCi}} \qquad \frac{p \vee q \;\parallel}{p,\, q \;\parallel}\vee_3^{\mathbf{mCi}}$$

$$\frac{\parallel}{p,\, q \;\parallel\; p \vee q}\vee_4^{\mathbf{mCi}} \qquad \frac{p \vee q,\, \circ(p \vee q) \;\parallel\; p \vee q}{\parallel}\vee_5^{\mathbf{mCi}}$$

$$\frac{\circ p \;\parallel}{\parallel\; \circ p}\circ_1^{\mathbf{mCi}} \qquad \frac{\parallel}{\circ\circ p \;\parallel}\circ_2^{\mathbf{mCi}} \qquad \frac{\parallel\; \circ p}{\circ p \;\parallel}\circ_3^{\mathbf{mCi}}$$

$$\frac{\parallel}{\circ p \;\parallel\; p}\circ_4^{\mathbf{mCi}} \qquad \frac{\parallel}{p \;\parallel\; \circ p}\circ_5^{\mathbf{mCi}}$$

$$\frac{\parallel}{\parallel\; \neg p,\, p}\neg_1^{\mathbf{mCi}} \qquad \frac{\neg p,\, \circ p,\, p \;\parallel}{\parallel}\neg_2^{\mathbf{mCi}} \qquad \frac{\neg p,\, p \;\parallel}{\parallel\; p}\neg_3^{\mathbf{mCi}}$$

$$\frac{\circ\neg p \;\parallel\; \neg p,\, p}{\parallel}\neg_4^{\mathbf{mCi}} \qquad \frac{\parallel\; \neg p,\, p}{\neg p \;\parallel}\neg_5^{\mathbf{mCi}} \qquad \frac{\parallel}{\neg p,\, \circ p \;\parallel}\neg_6^{\mathbf{mCi}}$$

$$\frac{\parallel}{\neg p,\, p \;\parallel}\neg_7^{\mathbf{mCi}} \qquad \frac{\parallel}{\circ\neg p \;\parallel\; p}\neg_8^{\mathbf{mCi}}$$

Note that the set of rules $\{©_i^{\mathbf{mCi}} \mid © \in \{\wedge, \vee, \supset\}, i \in \{1, 2, 3\}\}$ makes it clear that in the t-aspect of the induced B-consequence relation inhabits a logic extending positive classical logic, while the remaining rules for these connectives involve interactions between the two dimensions. Also, rule $\neg_2^{\mathbf{mCi}}$ indicates that $\circ$ satisfies one of the conditions for being considered a consistency connective in the logic inhabiting the t-aspect. In fact, all these observations are aligned with the fact that the logic inhabiting the t-aspect of $\vdots\!\mid\!\vdots_{\mathfrak{R}_{\mathbf{mCi}}}$ is precisely **mCi**.

7. Final remarks

In this work, we presented a Hilbert-style formalism for B-consequence relations by generalizing the Set-Set (or "multiple-conclusion") Hilbert-style formalism introduced in [57]. We also provided a proof-search and countermodel-search algorithm for finite and analytic systems based on [17], which runs in exponential time in general and in polynomial time when every rule of inference of the system at hand has a single formula in the succedent. Moreover, as a generalization of [43], we presented an axiomatization procedure that delivers (finite) analytic systems for the very inclusive class of (finite) sufficiently expressive partial non-deterministic B-matrices. We closed, then, with a new way of putting two one-dimensional consequence relations together by merging their characterizing matrices into a nd-B-matrix, showing, furthermore, that this operation may deliver finite two-dimensional axiomatizations for logics that are non-finitely axiomatizable in one dimension. The latter was mainly illustrated via the logic of formal inconsistency called **mCi**.

We highlight that our two-dimensional proof-formalism differs in important respects from the many-placed sequent calculi used in [5] to axiomatize (one-dimensional total) non-deterministic matrices (requiring no sufficient expressiveness) and in [34] for approaching multilateralism. First, a many-placed sequent calculus is not Hilbert-style: rules manipulate complex objects whose structure involve contexts and considerably deviate from the shape of the consequence relation being captured; our systems, on the other hand, are contained in their corresponding B-consequence relations. Second, when axiomatizing a matrix, the structure of many-placed sequents grows according to the number of values (n places for n truth-values); our rule schemas, in turn, remain with four places, and reflect the complexity of the underlying semantics in the complexity of the formulas being manipulated. Moreover, the study of many-placed sequents currently contemplates only one-dimensional consequence relations. Extending them to the two-dimensional case is a line of research worth exploring.

As opportunities of further future work, we envisage a more deep investigation of the other one-dimensional aspects — like q and p — of the B-consequence relations studied in this work; a study on how to convert the two-dimensional H-systems produced by our algorithm to n-place sequents or ordinary sequent calculi; and an investigation of the conditions under which a two-dimensional H-system can be converted into a one-dimensional H-system for each of its one-dimensional (Tarskian) aspects, and vice-versa, preserving desirable properties. In this respect, note that the examples presented in Chapter 6 allow us to eliminate the suspicion that a two-dimensional H-system $\mathfrak{R}$ may always be converted into SET-SET H-systems for the logics inhabiting the one-dimensional aspects of $\div|\div_{\mathfrak{R}}$ without losing the finiteness of the presentation.

Furthermore, we see as a promising line of investigation the generalization of the two-dimensional notion of consequence relation by allowing logics over different languages ([35]) — for instance, conflating different logics or different fragments of some given logic of interest — to coinhabit the same logical structure, each one along its own dimension, while controlling their interaction at the object-language level, taking advantage of the framework and the results in [44]. This opens the doors for a line of research on whether or to what extent the individual characteristics of these ingredient logics, such as their decidability status, may be preserved.

With respect to our proof-search algorithm, an important research path to be explored would involve comparing it with other proof-search algorithms for non-classical propositional logics (for instance, the theorem prover in [49] that encompasses **mbC** and **mCi**), and work on the design of heuristics for smarter choices of rule instances used to expand nodes during the search, as this may improve the performance of the algorithm on certain classes of logics.

At last, we also expect the present research to be extended so as to cover multidimensional notions of consequence, in order to provide increasingly general technical and philosophical grounds for the study of logical pluralism [9]. Note that the general account of symmetrical H-systems developed in detail in Chapter 3 already covers the possibility of having n-tuples of sets of formulas as antecedents and succedents in rules of inference, as well as in labels of rooted trees. In other words, it already makes sense to talk about $\textsc{Set}^n$-$\textsc{Set}^n$ H-systems, even though the associated notion of consequence relation remains to be developed and investigated. Actually, the extent to which the quite general derivation structures described in the mentioned chapter can be explored is still to be studied. For example, the fact that node labels may come from an arbitrary power set seems to provide suitable grounds for the generalization of fixed-point theorems and inductively defined sets, taking into account the possibility of reaching a fixed-point (or obtaining the desired objects) in each derivation branch.

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A. Implementation of the axiomatization and proof-search algorithms

In Chapter 5, we presented a recipe for axiomatizing any sufficiently expressive nd-B-matrix $\mathfrak{M}$, which results in a finite analytic $\textsc{Set}^2$-$\textsc{Set}^2$ H-system for $\mathfrak{M}$ whenever $\mathfrak{M}$ is finite. We have implemented this algorithm in the language `C++` and provided a command-line interface for it, which accepts a description in a `YAML` file and produces the resulting system in plain text in the terminal, in LaTeX code or as another `YAML` file, according to our choice. We may also choose what kind of simplification procedures should be executed over the raw system resulting from the Definition 81. In this appendix, we briefly explain how to describe, in an `YAML` file, the nd-B-matrix of interest and some details about the behaviour of the program under this input, as well as how to use the command-line interface to produce the desired axiomatization. We also show how to perform proof search over the generated analytic proof systems. The source code of the implementation is available at `https://github.com/greati/logicantsy`.

Building the command-line program

In order to avoid any compilation problem that might appear due to the operating system being used, we provide all that is necessary to build a Docker image and run the program from it. Visit docker.com for instructions on how to install and use Docker in each operating system. The building instructions are just the following, assuming Docker is already installed:

1. Clone the `git` repository:

   `git clone https://github.com/greati/logicantsy`

2. Run the following commands:

   `cd logicantsy`

   `docker build -t logicantsy .`

3. Test if all is good by running:

   `docker run logicantsy ./ltsy --help`

If the running instructions of the command-line interface are shown, then the build process was successful.

Building in Linux, no Docker

Those interested in building the tool in a Linux-based environment without using Docker, here are the instructions:

1. First of all, install the following packages/programs:

   - `gcc` (supporting `C++17`)
   - `git`
   - `make`
   - `cmake` ≥ 3.13
   - `flex`

- `bison`
- `boost`

2. Clone the `git` repository:

   `git clone https://github.com/greati/logicantsy`

3. Run the following commands:

   `cd logicantsy`

   `cmake -B build -S sources`

   `cd build`

   `make`

4. Test if all is good by running:

   `./ltsy --help`

Describing the nd-B-matrix

Below is an input file that produces an axiomatization for the nd-B-matrix of Example 82. We explain in the sequel each of its parts.

```
pnmatrix:
  values: [f, u, t]
  distinguished_sets:
    - [t, u]
    - [f, u]
  interpretation:
    p -> q:
      default: [t,u]
      restrictions:
        - [_, u]: []
        - [u, _]: []
        - [t, f]: [f]
    p and q:
      default: [f]
      restrictions:
        - [_, u]: []
        - [u, _]: []
        - [t, t]: [t,u]
```

```
    neg p:
      restrictions:
        - [f]: [t, u]
        - [u]: []
        - [t]: [f]
discriminator:
  f: [ [],  [p], [],  []  ]
  u: [ [p], [],  [p], []  ]
  t: [ [p], [],  [],  [p] ]
simplify_overlap: true
simplify_dilution: true
simplify_by_cuts: true
simplify_subrules_deriv: 10
simplify_derivation: 10
derive:
  r1: [[q],[p],[p -> q],[]]
  r2: [[],[],[p,q],[p and q]]
latex:
  "->": "\\to"
  and: "\\land"
  or: "\\lor"
  p: "\\varphi"
  p1: "\\varphi"
  p2: "\\psi"
  p3: "\\phi"
  p4: "\\sigma"
  q: "\\psi"
  f: "\\mathbf{f}"
  t: "\\mathbf{t}"
  neg: "\\neg"
```

The first section in this file, called `pnmatrix`, describes the nd-B-matrix to be axiomatized. It has three subsections:

- `values`: the list of truth-values. You may use any alphanumeric string for the name of a value. Internally, they become natural numbers in the order they are listed. In the present example, the set of values is $\{\mathbf{f}, \mathbf{u}, \mathbf{t}\}$.
- `distinguished_sets`: a list of the distinguished sets of the matrix being described. Our intention in the above file is to specify that $\mathsf{Y} := \{\mathbf{t}, \mathbf{u}\}$ and $\mathsf{N} := \{\mathbf{f}, \mathbf{u}\}$. Internally we store a list of sets, where the first element (of index 0) is Y, the second is ⅄, the third is N and the fourth is И.

- `interpretation`: a list of truth tables. The description of each truth table has the following elements:
  - a formula expressing the general form of a compound of the connective being interpreted. For example, by writing `p -> q`, we indicate that what follows is a description of an interpretation for `->`. We have some default connectives implemented, namely `neg` (unary) and `and`, `or`, `->` (binary). For introducing new connectives, use the notation `symbol(p1,p2,...,pm)`, where `m` is the arity of the connective. For example, `o(p)` would be a new unary connective called `o`, and `pt(p,q,r)` would be a ternary connective called `pt`.
  - `default`: an optional field providing a default output. This is useful when a truth table has the same output for many entries.
  - `restrictions`: a list indicating outputs for specific entries. The format of each element is `[v1,...,vm]:[o1,...,on]`, where `m` is the arity of the connective. We allow for the use of the underscore as a wildcard to indicate that for every value in that position we want the indicated output. For example, by writing `[_,u]`, we want to affect entries with inputs $(\mathbf{f}, \mathbf{u})$, $(\mathbf{u}, \mathbf{u})$ and $(\mathbf{t}, \mathbf{u})$. Internally, each restriction is processed top-down, and restrictions at upper positions are overwritten by ones in lower positions.

After describing the matrix, we must provide a discriminator for it, since the algorithm requires sufficient expressiveness. We do this in the section `discriminator`, and the format of the input is just like the table provided in Example 82: one row per value, and one column per each set of distinguished values. Each entry contains the set of separators for the corresponding value and set of distinguished values.

The next fields deal with the simplification options for the produced H-system (see Propositions 91, 93 and 94):

- `simplify_overlap`: activates the simplification by removing instances of overlap.

- `simplify_dilution`: activates the simplification by removing instances of dilution.
- `simplify_by_cuts`: activates the simplification by cut for formulas.
- `simplify_subrules_deriv`: optional field indicating the maximum depth of a derivation when trying to simplify by deriving subrules.
- `simplify_derivation`: optional field indicating the maximum number of rules allowed to be removed when simplifying the calculus by attempting to derive one rule from the others. This value is useful when the simplification is taking too long because the calculus may have gotten to a maximal level of simplification. In that case, reducing the value will make the process terminate earlier. The output in the terminal helps to see when that happens.

The next section, called `derive`, is totally optional and allows us to specify a list of statements of interest, and, for each of them, the program will run the proof-search algorithm described in Algorithm 4.1, printing in the terminal the produced tree, be it a proof or a failed attempt.

Finally, we may provide LaTeX translations of the symbols produced by the algorithm in section `latex`. This is another section that is not modified very often, unless new symbols are added to the description of the nd-B-matrix.

We close by showing another example of input file, this time for the 5-valued nd-B-matrix of Example 20 and Chapter 6:

```
pnmatrix:
  values: [I, T, F, t, f]
  distinguished_sets:
    - [I, T, t]
    - [I, T, f]
  interpretation:
    p or q:
      default: [f]
      restrictions:
        - [I, _]: [t, I]
        - [T, _]: [t, I]
        - [t, _]: [t, I]
        - [_, I]: [t, I]
```

```
      - [_, T]: [t, I]
      - [_, t]: [t, I]
  p and q:
    default: [t, I]
    restrictions:
      - [F, _]: [f]
      - [f, _]: [f]
      - [_, F]: [f]
      - [_, f]: [f]
  p -> q:
    restrictions:
      - [F, _]: [t, I]
      - [f, _]: [t, I]
      - [_, I]: [t, I]
      - [_, T]: [t, I]
      - [_, t]: [t, I]
      - [I, F]: [f]
      - [I, f]: [f]
      - [T, F]: [f]
      - [T, f]: [f]
      - [t, F]: [f]
      - [t, f]: [f]
  neg p:
    restrictions:
      - [T]: [F]
      - [F]: [T]
      - [t]: [f]
      - [f]: [t, I]
      - [I]: [t, I]
  o(p):
    default: [T]
    restrictions:
      - [I]: [F]
discriminator:
  I: [  [p],        [o(p)], [p],   []  ]
  T: [  [o(p),p],   [],     [p],   []  ]
  F: [  [],         [p],    [],    [p] ]
  t: [  [p],        [],     [],    [p] ]
  f: [  [],         [p],    [p],   []  ]
simplify_overlap: true
simplify_dilution: true
simplify_by_cuts: true
simplify_subrules_deriv: 9
simplify_derivation: 8
latex:
  "->": "\\to"
```

```
  p: "\\varphi"
  p1: "\\varphi"
  p2: "\\psi"
  p3: "\\phi"
  p4: "\\sigma"
  q: "\\psi"
  r: "\\phi"
  o: "\\circ"
  and: "\\land"
  or: "\\lor"
  f: "\\mathbf{f}"
  t: "\\mathbf{t}"
  neg: "\\neg"
```

Running the program

Assume that the `YAML` file is named `/path/to/bmatrix-description.yml`. In order to run the algorithm on this input, just execute the following command in the terminal (assuming Docker is being used):

```
docker run -v /path/to/:/input logicantsy \
./ltsy axiomatize-monadic-matrix -f /input/bmatrix-example.yml \
-o latex -s /input/result.tex
```

This produces the `result.tex` file in the folder `/path/to`. We may then compile it to a `PDF` file as usual. Another possibility is to output the result in the terminal directly:

```
docker run -v /path/to/:/input logicantsy \
./ltsy axiomatize-monadic-matrix -f /input/bmatrix-example.yml -o plain
```

A third possibility is to output the resulting proof system in a new `yaml` file, so that it can be used to run the proof-search algorithm multiple times without the need to regenerate the proof system every time. The command for this is:

```
docker run -v /path/to/:/input logicantsy \
./ltsy axiomatize-monadic-matrix -f /input/bmatrix-example.yml \
-o yaml -s /input/result.yaml
```

See the next subsection to better understand the generated file and to see how to employ it in performing proof search over the proof system it describes.

You may find examples of outputs in the repository, under the directory `examples/outputs`.

Proof search using a generated `YAML` file

The `YAML` file that the axiomatization tool generates looks like the following:

```yaml
calculus:
  r2: [[],[],["p1 and p2"],["p1", "p2"]]
  r6: [[],[],["p1"],["p1 and p2"]]
  r8: [[],[],["p2"],["p1 and p2"]]
  r12: [["p1 and p2"],["p1"],[],[]]
  r13: [["p1 and p2"],["p2"],[],[]]
  r19: [["p1", "p2"],["p1 and p2"],[],[]]
  r1: [[],[],[],["bot"]]
  r14: [["bot"],[],[],[]]
  r9: [[],["neg p1"],["p1"],[]]
  r11: [[],["p1"],["neg p1"],[]]
  r15: [["neg p1"],[],[],["p1"]]
  r17: [["p1"],[],[],["neg p1"]]
  r3: [[],[],["p1 or p2"],["p1"]]
  r4: [[],[],["p1 or p2"],["p2"]]
  r7: [[],[],["p1", "p2"],["p1 or p2"]]
  r16: [["p1 or p2"],["p1", "p2"],[],[]]
  r18: [["p1"],["p1 or p2"],[],[]]
  r20: [["p2"],["p1 or p2"],[],[]]
  r5: [[],[],["top"],[]]
  r10: [[],["top"],[],[]]
analyticity_formulas: ["p"]
simplify_overlap: false
simplify_dilution: false
simplify_by_cuts: false
simplify_by_subrule_deriv: 0
prem_conc_correspondence: [[0,1],[2,3]]
sequent_dset_correspondence: [0,1,2,3]
simplify_max_level: 0
derive:
  somerule: [["p2"],["p1 or p2"],[],[]]
```

The generated proof system is described in the `calculus` section. All the other attributes are there because this same input file can be used as input to another tool that performs the proof search over that calculus, in order to derive the statements described in the `derive` section. Attributes related to simplification can be set up just as in the input file for the axiomatization procedure. The effect is that the simplification will be performed right before the proof search starts. This is how this tool can be used:

```
docker run -v /path/to/:/input logicantsy \
./ltsy analytically-derive -f /input/generated-calculus.yaml
```

The one-dimensional algorithm is also implemented

We should point out that this implementation also works for sufficiently expressive one-dimensional matrices to produce analytic finite SET-SET H-systems, according to the algorithm of [17]. The input file follows the same structure detailed before, we just need to provide this time a single set of distinguished truth values and change the format of the discriminator to have only two columns. Here is an example of an YAML file for generating a finite analytic SET-SET H-system for the logic FDE (the one in Example 47, but now containing also nullary connectives $\top$ and $\bot$):

```yaml
pnmatrix:
  values: [bot, t, f, top]
  distinguished_sets:
    - [top, t]
  interpretation:
    p and q:
      restrictions:
        - [f, _]: [f]
        - [_, f]: [f]
        - [top, t]: [top]
        - [bot, t]: [bot]
        - [t, top]: [top]
        - [t, bot]: [bot]
        - [top, bot]: [f]
        - [bot, top]: [f]
        - [top,top]: [top]
        - [bot,bot]: [bot]
        - [t, t]: [t]
    p or q:
      restrictions:
        - [t, _]: [t]
        - [_, t]: [t]
        - [top, f]: [top]
```

```
        - [bot, f]: [bot]
        - [f, top]: [top]
        - [f, bot]: [bot]
        - [top, bot]: [t]
        - [bot, top]: [t]
        - [top,top]: [top]
        - [bot,bot]: [bot]
        - [f, f]: [f]
    neg p:
      restrictions:
        - [t]: [f]
        - [f]: [t]
        - [bot]: [bot]
        - [top]: [top]
    bot():
      restrictions:
        - []: [f]
    top():
      restrictions:
        - []: [t]
discriminator:
  top:  [ [p, neg p], []          ]
  t:    [ [p],        [neg p]     ]
  f:    [ [neg p],    [p]         ]
  bot:  [ [],         [p, neg p] ]
simplify_overlap: true
simplify_dilution: true
simplify_by_cuts: true
simplify_derivation: 30
simplify_subrules_deriv: 30
latex:
  "->": "\\to"
  p: "\\varphi"
  p1: "\\varphi"
  p2: "\\psi"
  p3: "\\phi"
  q: "\\psi"
  r: "\\phi"
  and: "\\land"
  or: "\\lor"
  f: "\\mathbf{f}"
  t: "\\mathbf{t}"
  bot: "\\bot"
  top: "\\top"
  neg: "\\neg"
```