Algorithmic and Hardness Results for the Colorful Components Problems

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Problem(s) Definition(s)

Related work

Our Results

Minimum Singleton Vertices

Approximation algorithm for MEC

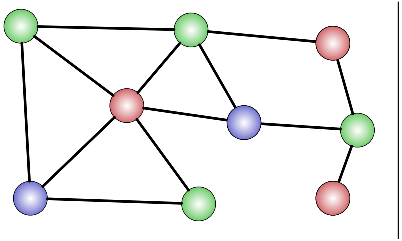
$|V|^{1/3-\varepsilon}$ hardness of approximation for MEC

Hardness of approximation for MEC for 3 colors

Open problems

Colorful Components framework

Input: An undirected graph G with colored vertices.
Goal: Delete edges to get *colorful components*.



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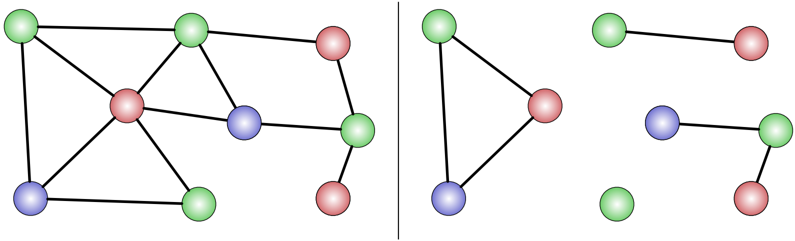
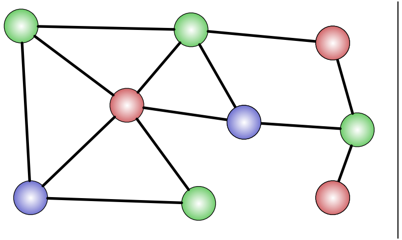


Minimum Singleton Vertices

Goal: Minimize the number of singleton vertices.



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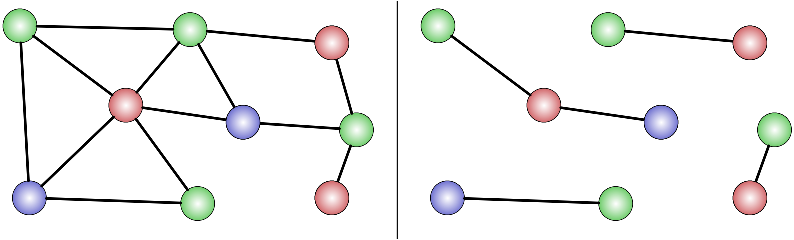
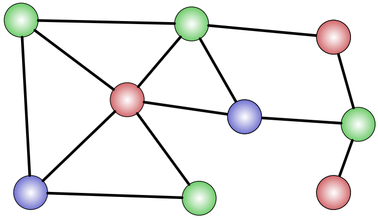


Maximum Edges in Transitive Closure

Goal: Maximize the number of edges in the transitive closure.



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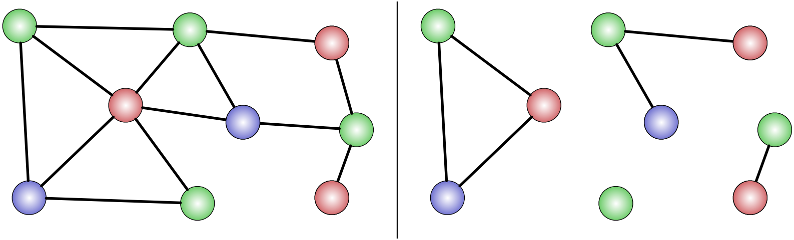

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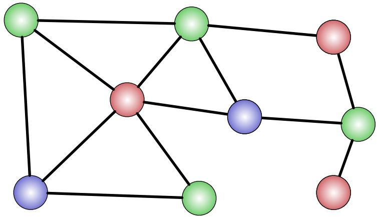

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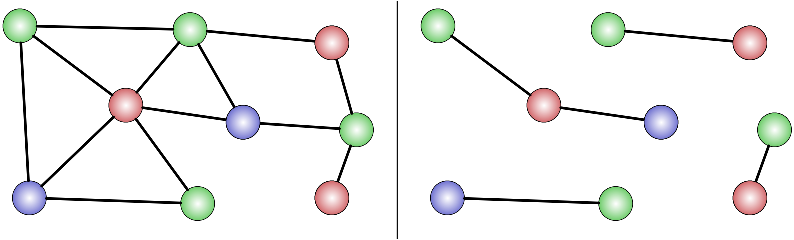

Related Work

Colorful Components asks for the minimum number of edge deletions.

- NP-hard for ≥ 3 colors + FPT algorithms [Bruckner, Hüffner, Komusiewicz, Niedermeier, Thiel, Uhlmann 2012]
- APX-hard
- Heuristics [Bruckner, Hüffner, Komusiewicz, Niedermeier, Thiel, Uhlmann 2012] and [Zheng, Swenson, Lyons, Sankoff 2011]
- Approximation algorithms for a special case [He, Liu, Zha 2000]
- $O(\log |C|)$ approximation (from Multi-Multiway Cut)

Related Work

- MSV and MEC are introduced by [Zheng, Swenson, Lyons, Sankoff 2011]
- Applications in comparative genomics
- Heuristics
- MSV and MEC are conjectured to be NP-hard
- 2/3 approximation for the dual of MSV (MAX-OREC) [Tremblay-Savard, Swenson 2012]

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3 MCC is hard to approximate within $|V|^{1/2-\epsilon}$
- Reduction from Minimum Clique Partition

A lower bound for MSV

Lemma

In any feasible solution, for any color c, the number of singletons is at least:

$$s_c = max(|V'| - |\bar{N}(V')|)$$

V' = *a set of vertices colored with c*
$\bar{N}(V')$ = *neighborhood with color* $\neq c$

A lower bound for MSV

Proof.

Fix a feasible partition G' and a color c.
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$\#$ non-singletons $\leq |\bar{N}(V')|$
$\#$ singletons is at least $|V'| - |\bar{N}(V')|$

□

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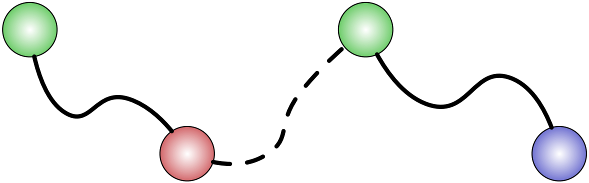
4. $\#$ singleton vertices of color c decreases
5. $\#$ other singletons does not increase
6. Each connected component of G' is a singleton, edge or star
7. When there is no alternating path, we match the lower bound

How to find alternating paths?

- Start with a singleton of col. c
- Connect a vertex of col. c with a vertex of col. $\neq c$ and viceversa
- Alternate edges in $E \setminus E'$ with edges in E'



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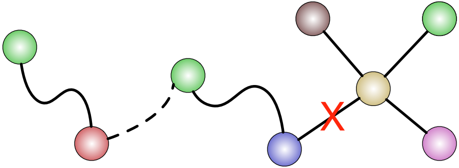

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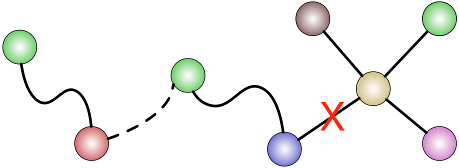
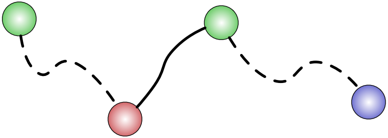
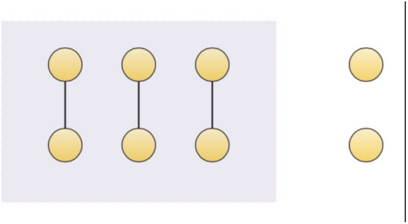


- No col. c in the component $\Rightarrow$ one singleton less + vertices of color c are “switched” between components



$\sqrt{2 \cdot \mathrm{OPT}}$-approximation



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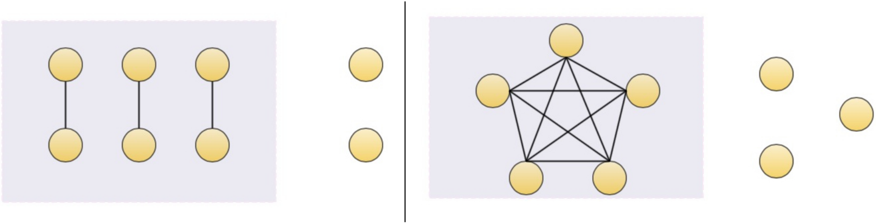

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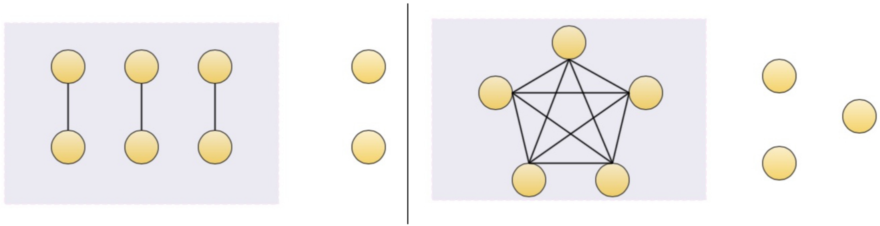
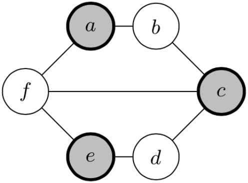


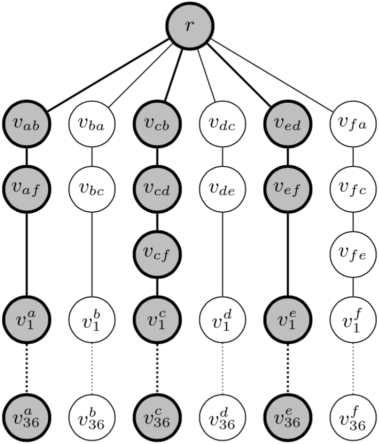
(Handwaving)

In the worst case: $\mathrm{OPT} = \binom{|V|}{2}$ and $\mathrm{MSV} = |V|/2$

Approximation ratio: $\frac{\binom{|V|}{2}}{\frac{|V|}{2}} = \sqrt{2 \cdot \mathrm{OPT}}$

$|V|^{1/3-\varepsilon}$ hardness of approximation for MEC





Hardness of approximating MEC

Lemma

IS of size $\alpha \Rightarrow$ MEC of value $\binom{\alpha n^2}{2}$.

Lemma

MEC of value $n^5/2 + \alpha^2 \cdot n^4 \Rightarrow$ IS of size α.

Theorem

It is NP-hard to approximate MEC within a factor of $|V|^{1/3-\varepsilon}$.

Proof.

NP-hard: IS $< n^\varepsilon$ or IS $\geq n^{1-\varepsilon}$?
$OPT_{MEC} \leq n^5/2 + n^{4+2\varepsilon}$
$OPT_{MEC} \geq n^{6-2\varepsilon}/2$
The number of vertices of G' is in $\Theta(n^3)$ □

APX-hardness of MEC for $|C| = 3$

Reduction from Max Bounded 3D-matching:
each element is in ≤ 3 triples.

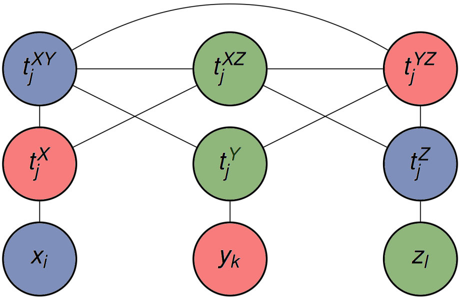


Figure : A subgraph corresponding to a triple $t_j = (x_i, y_k, z_l)$

APX-hardness of MEC for $|C| = 3$

All the vertices are matchend $\Rightarrow OPT_{MEC} = 6\cdot$ # triples + # elements

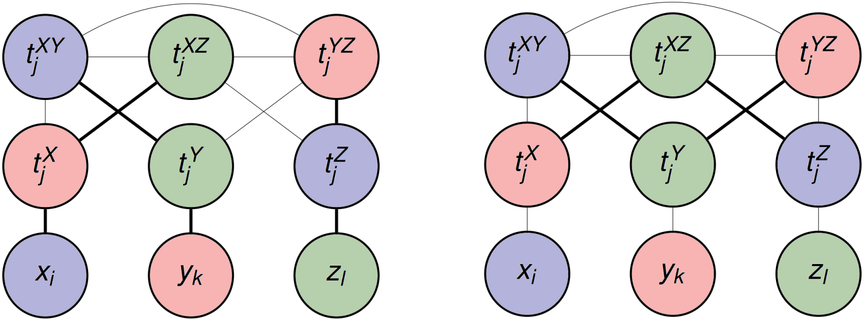


Figure : A triple from T' (left) and a triple from $T \setminus T'$ (right).

Future work

- Other meaningful objective functions.
- Test MSV algorithm on real data.
- Close the gap between the hardness and approximability of MEC.
- Maximizing the number of edges in the connected components.

Merci!

Dziękuję!

Спасибо!

Mulţumesc!

Thank you!