Structure in CSP and SAT
A case study

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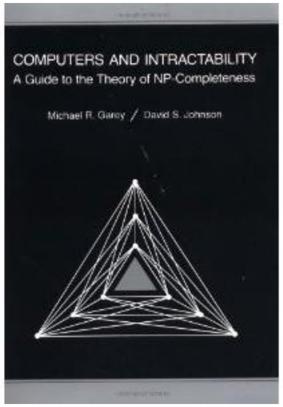
Joint work with Justyna Petke

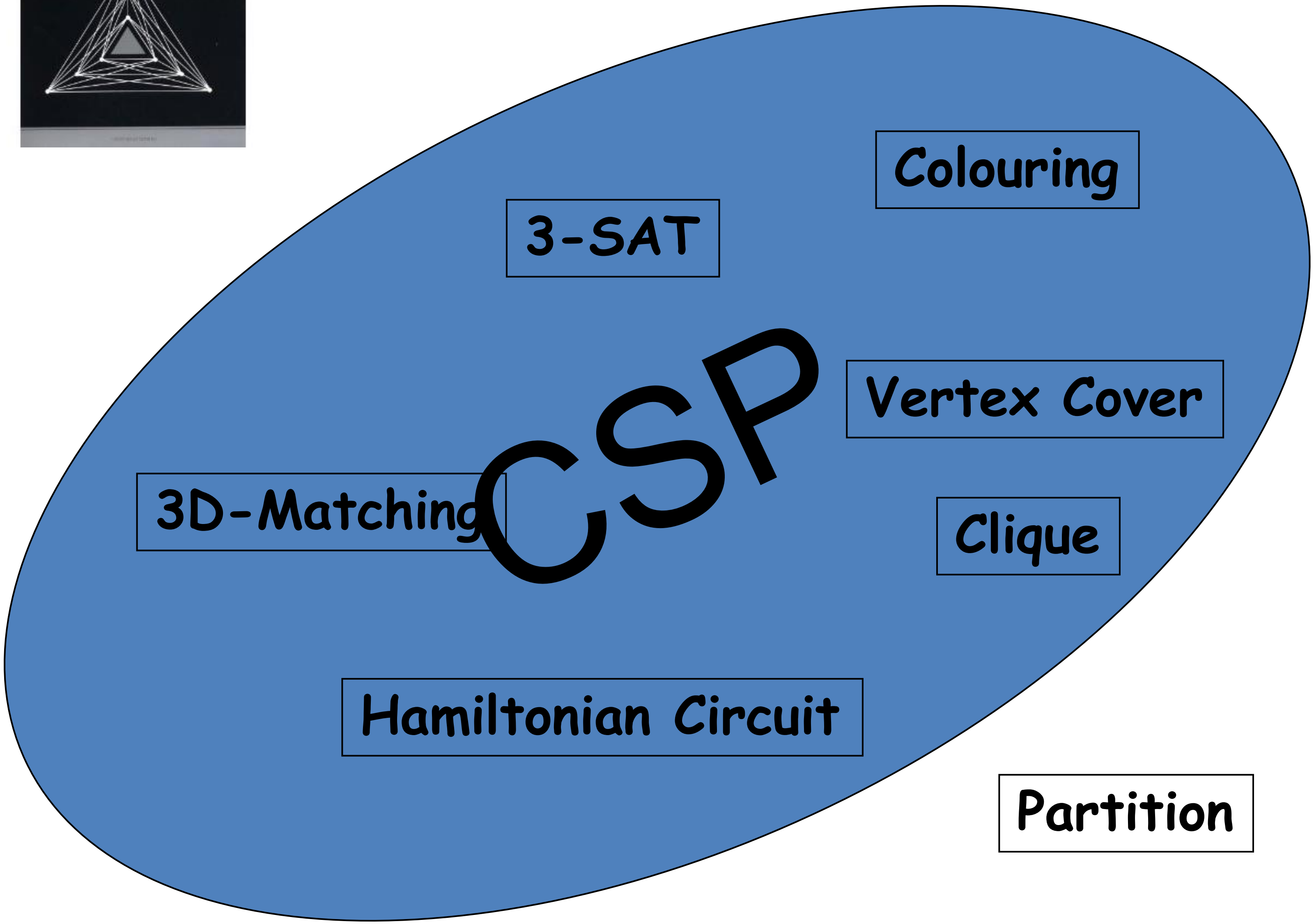
Outline

- The CSP as a general framework
- A family of examples with simple structure...

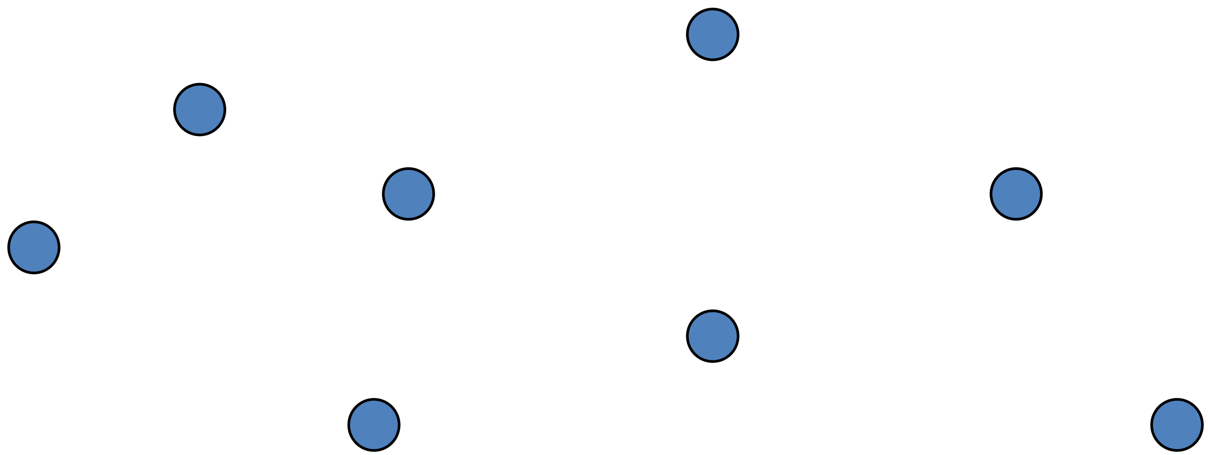
  ...that is surprisingly tricky
- Encoding into SAT
- Finding the structure
- Exploiting the structure
- Open questions…

“Basic” Problems





A General Framework



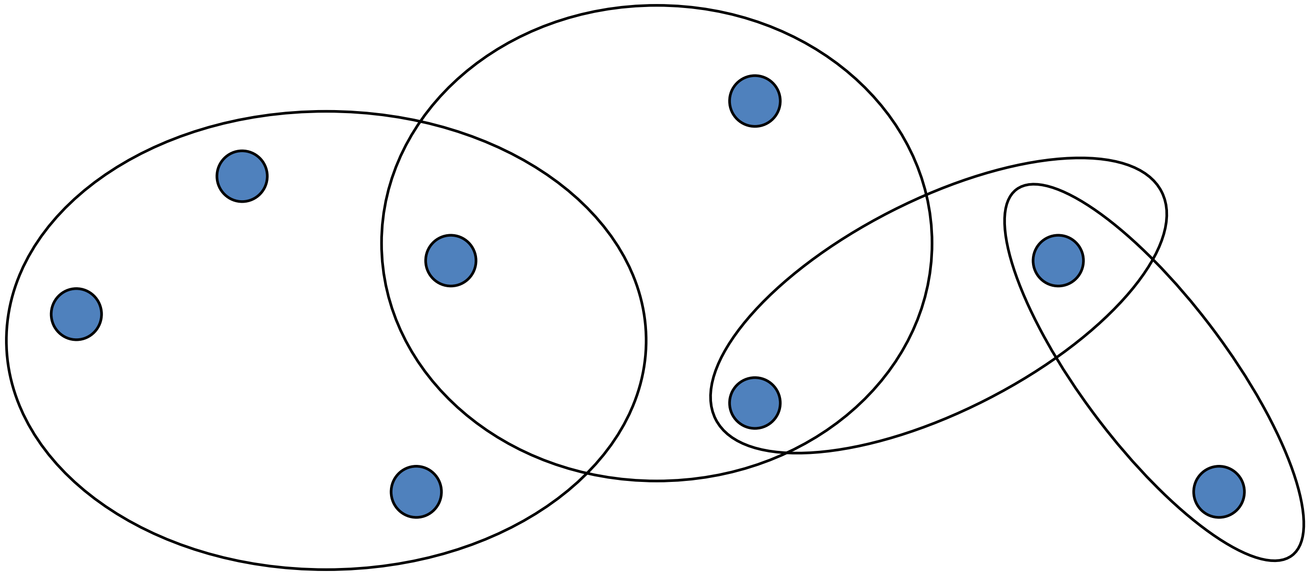
Variables ● = Talks to be scheduled at conference

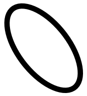
Transmitters to be assigned frequencies

Amino acids to be located in space

Circuit components to be placed on a chip

A General Framework



Constraints  = All invited talks on different days

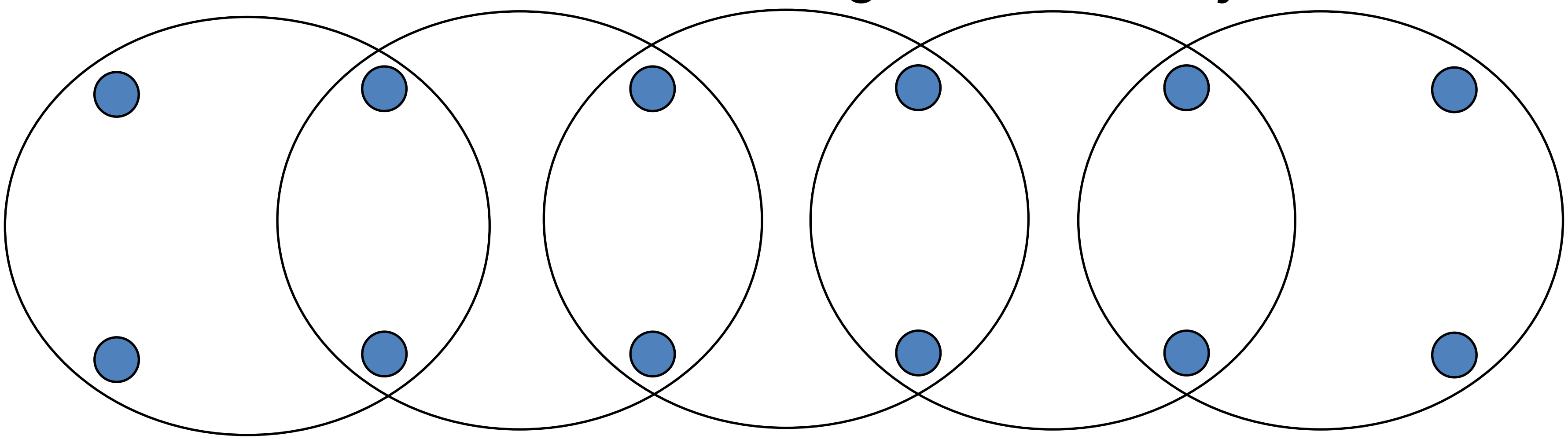
No interference between near transmitters

$x + y + z > 0$

At least 1μm between wires

A Family of Examples

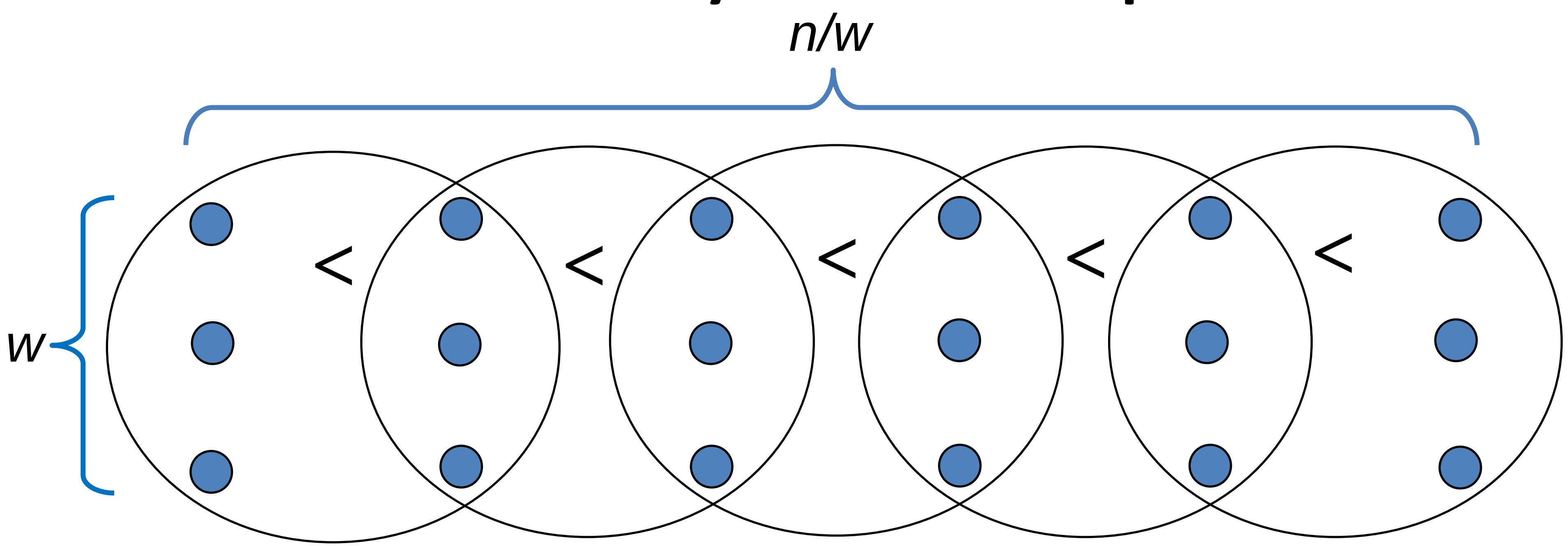
$e + f < g + h$ $\quad$ $i + j < k + l$



$c + d < e + f$ $\quad$ $g + h < i + j$

Constraints = $a + b < c + d$

A Family of Examples



n variables

width: *w*

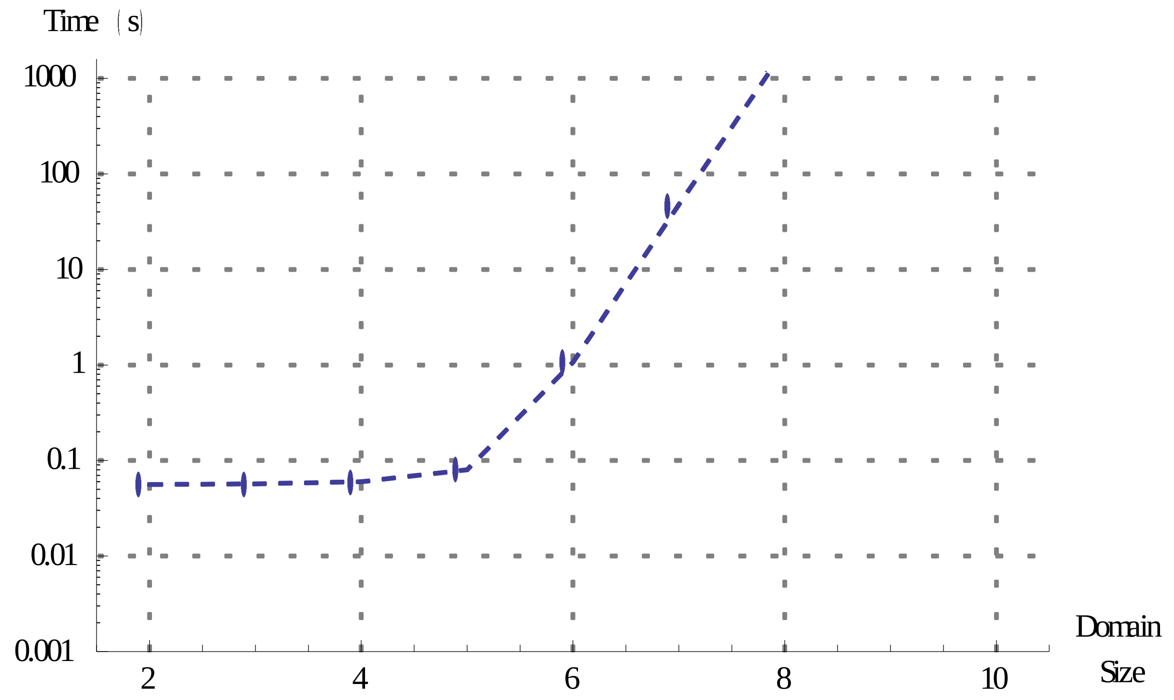
domain: *{1,2,..,d}*

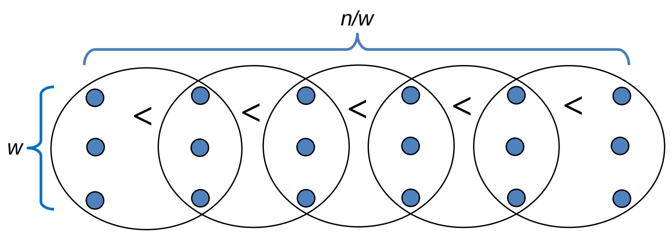
```
array[1..4] of var 1..2:X1;
array[1..4] of var 1..2:X2;
constraint
forall(i in 1..3)(
   X1[i]+X2[i] < X1[i+1]+X2[i+1]);
solve satisfy;
```

Specification in MiniZinc for *w=2, d=2*

A Family of Examples

Solution by constraint solver Minion





n variables

width: *w*

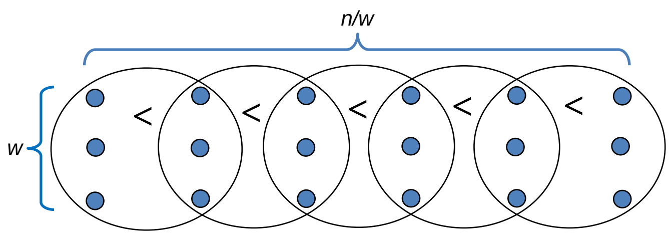
domain: *{1,2,..,d}*

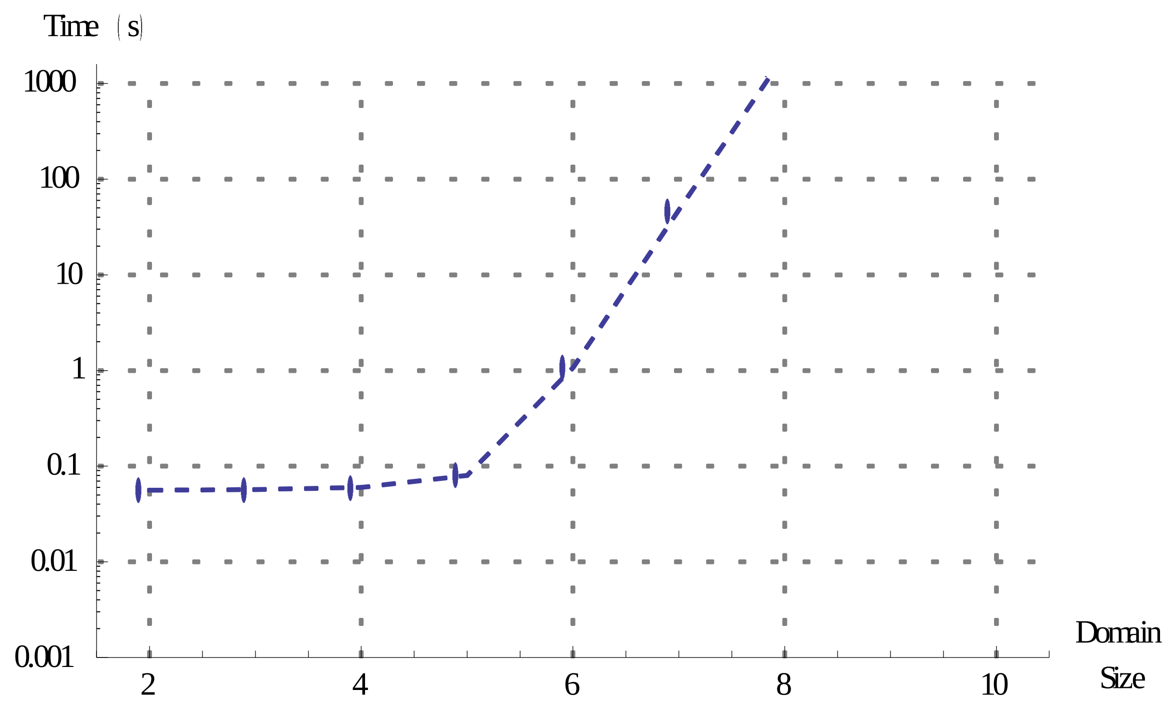
```
array[1..2d] of var 1..d:X1;
array[1..2d] of var 1..d:X2;
constraint
forall(i in 1..2d-1)(
   X1[i]+X2[i] < X1[i+1]+X2[i+1]);
solve satisfy;
```

Specification in MiniZinc for *w=2*

Lesson I

Solution by constraint solver Minion



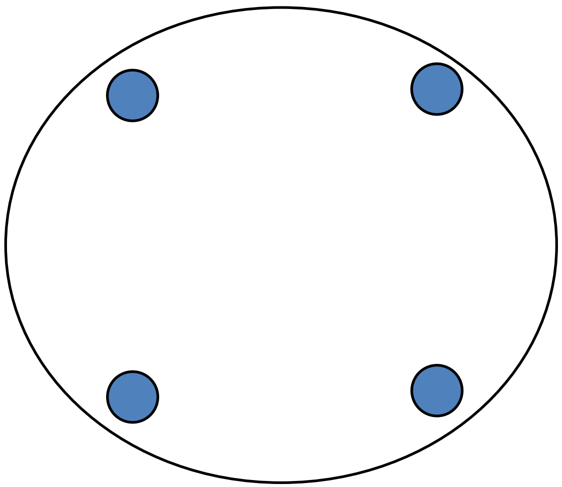


Simple structure is not always exploited even by sophisticated solvers

TRY A DIFFERENT MODEL…

Direct Encoding

$a + b < c + d$

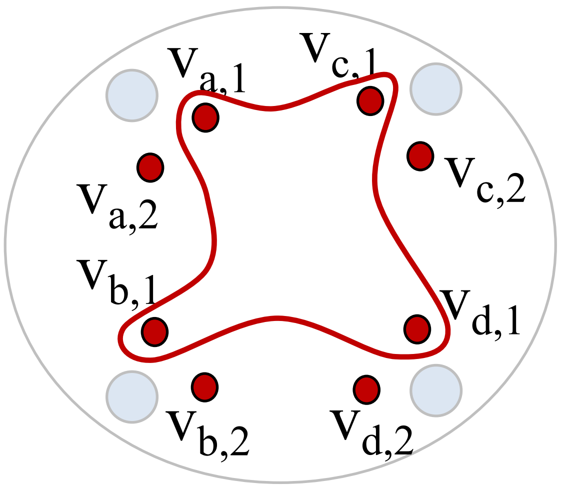


domain: *{1,2}*

a	b	c	d	
1	1	1	1	✗
1	1	1	2	✓
1	1	2	1	✓
1	1	2	2	✓
1	2	1	1	✗
1	2	1	2	✗
1	2	2	1	✗
1	2	2	2	✓
2	1	1	1	✗
2	1	1	2	✗
2	1	2	1	✗
2	1	2	2	✓
2	2	1	1	✗
2	2	1	2	✗
2	2	2	1	✗
2	2	2	2	✗

Direct Encoding

a + b < c + d



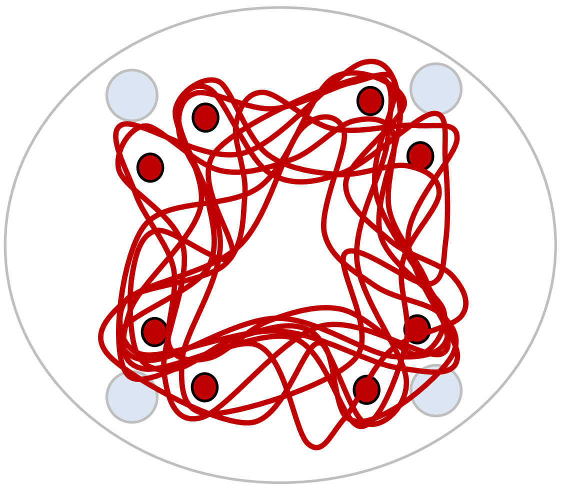
domain: *{1,2}*

a	b	c	d	
1	1	1	1	✗
1	1	1	2	✓
1	1	2	1	✓
1	1	2	2	✓
1	2	1	1	✗
1	2	1	2	✗
1	2	2	1	✗
1	2	2	2	✓
2	1	1	1	✗
2	1	1	2	✗
2	1	2	1	✗
2	1	2	2	✓
2	2	1	1	✗
2	2	1	2	✗
2	2	2	1	✗
2	2	2	2	✗

$(\neg v_{a,1} \vee \neg v_{b,1} \vee \neg v_{c,1} \vee \neg v_{d,1})$

Direct Encoding

$a + b < c + d$



domain: *{1,2}*

a	b	c	d	
1	1	1	1	✗
1	1	1	2	✓
1	1	2	1	✓
1	1	2	2	✓
1	2	1	1	✗
1	2	1	2	✗
1	2	2	1	✗
1	2	2	2	✓
2	1	1	1	✗
2	1	1	2	✗
2	1	2	1	✗
2	1	2	2	✓
2	2	1	1	✗
2	2	1	2	✗
2	2	2	1	✗
2	2	2	2	✗

$$(\neg v_{a,1} \vee \neg v_{b,1} \vee \neg v_{c,1} \vee \neg v_{d,1})$$

$$\wedge (\neg v_{a,1} \vee \neg v_{b,2} \vee \neg v_{c,1} \vee \neg v_{d,1})$$
$$\wedge (\neg v_{a,1} \vee \neg v_{b,2} \vee \neg v_{c,1} \vee \neg v_{d,2})$$
$$\wedge (\neg v_{a,1} \vee \neg v_{b,2} \vee \neg v_{c,2} \vee \neg v_{d,1})$$

$$\wedge (\neg v_{a,2} \vee \neg v_{b,1} \vee \neg v_{c,1} \vee \neg v_{d,1})$$
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$$\wedge (\neg v_{a,2} \vee \neg v_{b,2} \vee \neg v_{c,1} \vee \neg v_{d,2})$$
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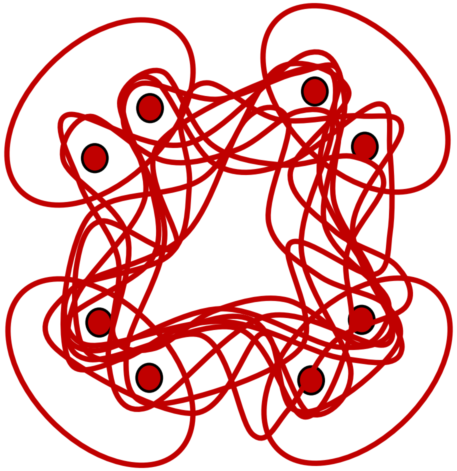
$$\wedge (\neg v_{a,2} \vee \neg v_{b,2} \vee \neg v_{c,2} \vee \neg v_{d,2})$$

Direct Encoding



domain: *{1,2}*

$$(v_{a,1} \lor v_{a,2})$$
$$\land (v_{b,1} \lor v_{b,2})$$
$$\land (v_{c,1} \lor v_{c,2})$$
$$\land (v_{d,1} \lor v_{d,2})$$

$$(\neg v_{a,1} \lor \neg v_{b,1} \lor \neg v_{c,1} \lor \neg v_{d,1})$$

$$\land (\neg v_{a,1} \lor \neg v_{b,2} \lor \neg v_{c,1} \lor \neg v_{d,1})$$
$$\land (\neg v_{a,1} \lor \neg v_{b,2} \lor \neg v_{c,1} \lor \neg v_{d,2})$$
$$\land (\neg v_{a,1} \lor \neg v_{b,2} \lor \neg v_{c,2} \lor \neg v_{d,1})$$

$$\land (\neg v_{a,2} \lor \neg v_{b,1} \lor \neg v_{c,1} \lor \neg v_{d,1})$$
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$$\land (\neg v_{a,2} \lor \neg v_{b,1} \lor \neg v_{c,2} \lor \neg v_{d,1})$$

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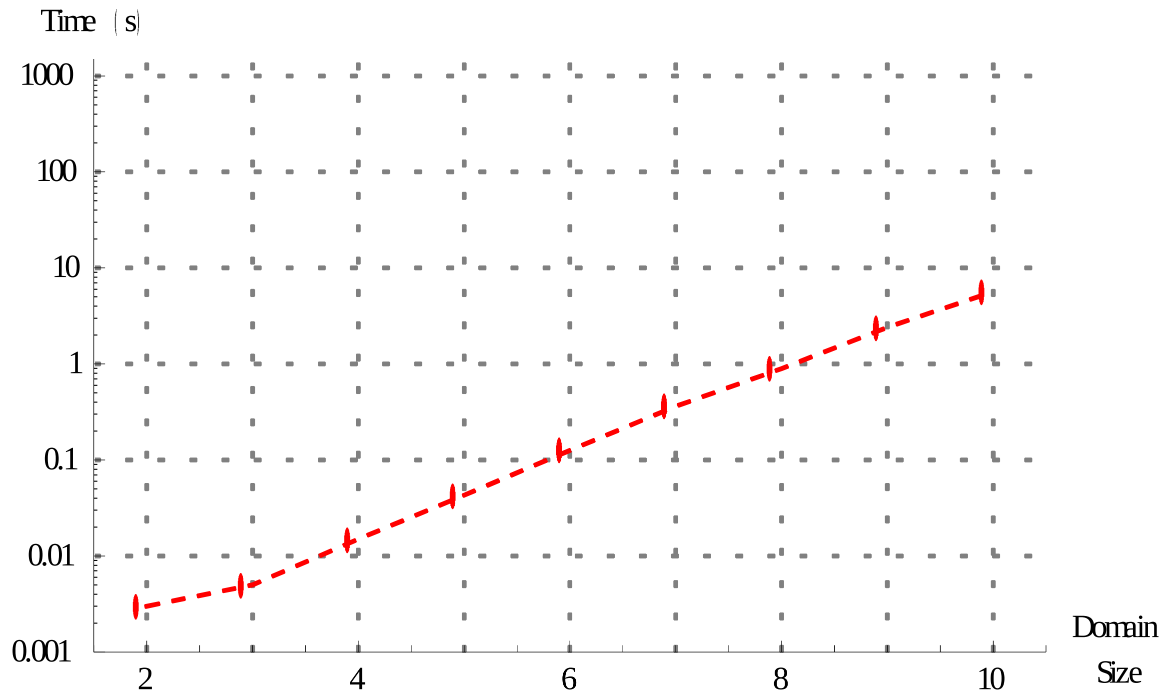
$$\land (\neg v_{a,2} \lor \neg v_{b,2} \lor \neg v_{c,1} \lor \neg v_{d,2})$$
$$\land (\neg v_{a,2} \lor \neg v_{b,2} \lor \neg v_{c,2} \lor \neg v_{d,1})$$
$$\land (\neg v_{a,2} \lor \neg v_{b,2} \lor \neg v_{c,2} \lor \neg v_{d,2})$$

Direct Encoding

Solution by
SAT solver
MiniSAT

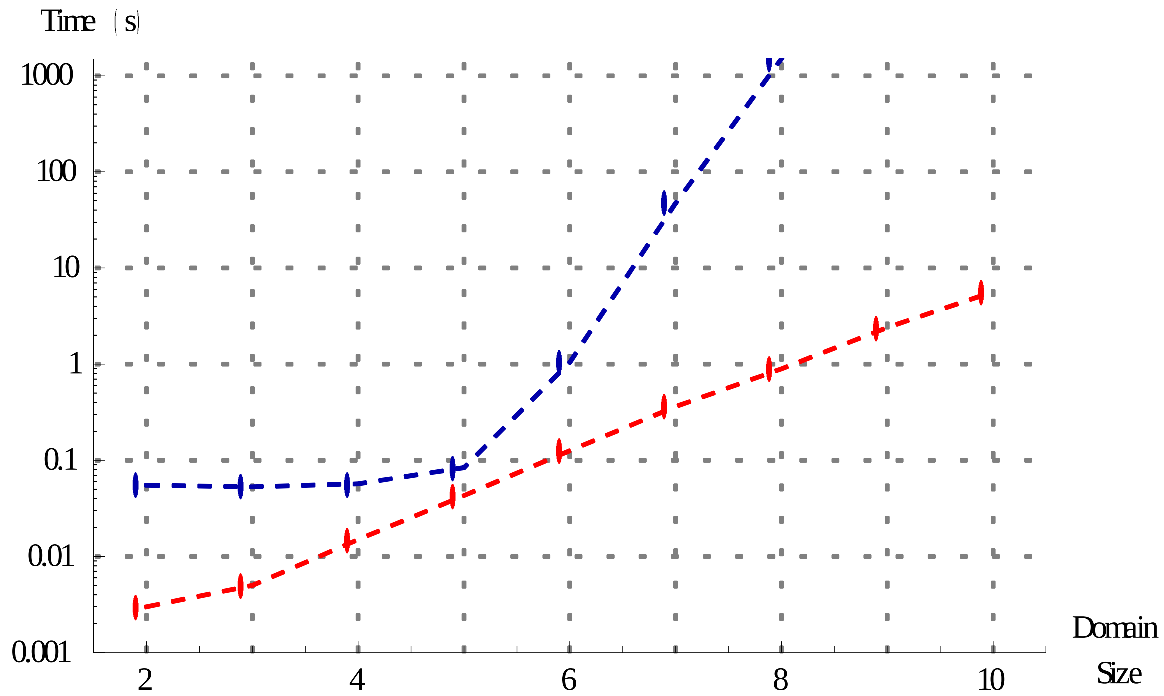


$$
\begin{gathered}
(v_{1,1} \lor v_{1,2}) \land (v_{2,1} \lor v_{2,2}) \land (v_{3,1} \lor v_{3,2}) \land (v_{4,1} \lor v_{4,2}) \land (v_{5,1} \lor v_{5,2}) \land (v_{6,1} \lor v_{6,2}) \land (v_{7,1} \lor v_{7,2}) \land (v_{8,1} \lor v_{8,2}) \\
\land (\neg v_{1,1} \lor \neg v_{2,1} \lor \neg v_{3,1} \lor \neg v_{4,1}) \land (\neg v_{1,1} \lor \neg v_{2,2} \lor \neg v_{3,1} \lor \neg v_{4,1}) \land (\neg v_{1,1} \lor \neg v_{2,2} \lor \neg v_{3,1} \lor \neg v_{4,2}) \\
\land (\neg v_{1,1} \lor \neg v_{2,2} \lor \neg v_{3,2} \lor \neg v_{4,1}) \land (\neg v_{1,2} \lor \neg v_{2,1} \lor \neg v_{3,1} \lor \neg v_{4,1}) \land (\neg v_{1,2} \lor \neg v_{2,1} \lor \neg v_{3,1} \lor \neg v_{4,2}) \\
\land (\neg v_{1,2} \lor \neg v_{2,1} \lor \neg v_{3,2} \lor \neg v_{4,1}) \land (\neg v_{1,2} \lor \neg v_{2,2} \lor \neg v_{3,1} \lor \neg v_{4,1}) \land (\neg v_{1,2} \lor \neg v_{2,2} \lor \neg v_{3,1} \lor \neg v_{4,2}) \\
\land (\neg v_{1,2} \lor \neg v_{2,2} \lor \neg v_{3,2} \lor \neg v_{4,1}) \land (\neg v_{1,2} \lor \neg v_{2,2} \lor \neg v_{3,2} \lor \neg v_{4,2}) \land (\neg v_{3,1} \lor \neg v_{4,1} \lor \neg v_{5,1} \lor \neg v_{6,1}) \\
\land (\neg v_{3,1} \lor \neg v_{4,2} \lor \neg v_{5,1} \lor \neg v_{6,1}) \land (\neg v_{3,1} \lor \neg v_{4,2} \lor \neg v_{5,1} \lor \neg v_{6,2}) \land (\neg v_{3,1} \lor \neg v_{4,2} \lor \neg v_{5,2} \lor \neg v_{6,1}) \\
\land (\neg v_{3,2} \lor \neg v_{4,1} \lor \neg v_{5,1} \lor \neg v_{6,1}) \land (\neg v_{3,2} \lor \neg v_{4,1} \lor \neg v_{5,1} \lor \neg v_{6,2}) \land (\neg v_{3,2} \lor \neg v_{4,1} \lor \neg v_{5,2} \lor \neg v_{6,1}) \\
\land (\neg v_{3,2} \lor \neg v_{4,2} \lor \neg v_{5,1} \lor \neg v_{6,1}) \land (\neg v_{3,2} \lor \neg v_{4,2} \lor \neg v_{5,1} \lor \neg v_{6,2}) \land (\neg v_{3,2} \lor \neg v_{4,2} \lor \neg v_{5,2} \lor \neg v_{6,1}) \\
\land (\neg v_{3,2} \lor \neg v_{4,2} \lor \neg v_{5,2} \lor \neg v_{6,2}) \land (\neg v_{5,1} \lor \neg v_{6,1} \lor \neg v_{7,1} \lor \neg v_{8,1}) \land (\neg v_{5,1} \lor \neg v_{6,2} \lor \neg v_{7,1} \lor \neg v_{8,1}) \\
\land (\neg v_{5,1} \lor \neg v_{6,2} \lor \neg v_{7,1} \lor \neg v_{8,2}) \land (\neg v_{5,1} \lor \neg v_{6,2} \lor \neg v_{7,2} \lor \neg v_{8,1}) \land (\neg v_{5,2} \lor \neg v_{6,1} \lor \neg v_{7,1} \lor \neg v_{8,1}) \\
\land (\neg v_{5,2} \lor \neg v_{6,1} \lor \neg v_{7,1} \lor \neg v_{8,2}) \land (\neg v_{5,2} \lor \neg v_{6,1} \lor \neg v_{7,2} \lor \neg v_{8,1}) \land (\neg v_{5,2} \lor \neg v_{6,2} \lor \neg v_{7,1} \lor \neg v_{8,1}) \\
\land (\neg v_{5,2} \lor \neg v_{6,2} \lor \neg v_{7,1} \lor \neg v_{8,2}) \land (\neg v_{5,2} \lor \neg v_{6,2} \lor \neg v_{7,2} \lor \neg v_{8,1}) \land (\neg v_{5,2} \lor \neg v_{6,2} \lor \neg v_{7,2} \lor \neg v_{8,2})
\end{gathered}
$$

Direct Encoding

Solution by
SAT solver
MiniSAT

compared to
Minion



$$(v_{1,1} \vee v_{1,2}) \wedge (v_{2,1} \vee v_{2,2}) \wedge (v_{3,1} \vee v_{3,2}) \wedge (v_{4,1} \vee v_{4,2}) \wedge (v_{5,1} \vee v_{5,2}) \wedge (v_{6,1} \vee v_{6,2}) \wedge (v_{7,1} \vee v_{7,2}) \wedge (v_{8,1} \vee v_{8,2})$$

$$\wedge (\neg v_{1,1} \vee \neg v_{2,1} \vee \neg v_{3,1} \vee \neg v_{4,1}) \wedge (\neg v_{1,1} \vee \neg v_{2,2} \vee \neg v_{3,1} \vee \neg v_{4,1}) \wedge (\neg v_{1,1} \vee \neg v_{2,2} \vee \neg v_{3,1} \vee \neg v_{4,2})$$

$$\wedge (\neg v_{1,1} \vee \neg v_{2,2} \vee \neg v_{3,2} \vee \neg v_{4,1}) \wedge (\neg v_{1,2} \vee \neg v_{2,1} \vee \neg v_{3,1} \vee \neg v_{4,1}) \wedge (\neg v_{1,2} \vee \neg v_{2,1} \vee \neg v_{3,1} \vee \neg v_{4,2})$$

$$\wedge (\neg v_{1,2} \vee \neg v_{2,1} \vee \neg v_{3,2} \vee \neg v_{4,1}) \wedge (\neg v_{1,2} \vee \neg v_{2,2} \vee \neg v_{3,1} \vee \neg v_{4,1}) \wedge (\neg v_{1,2} \vee \neg v_{2,2} \vee \neg v_{3,1} \vee \neg v_{4,2})$$

$$\wedge (\neg v_{1,2} \vee \neg v_{2,2} \vee \neg v_{3,2} \vee \neg v_{4,1}) \wedge (\neg v_{1,2} \vee \neg v_{2,2} \vee \neg v_{3,2} \vee \neg v_{4,2}) \wedge (\neg v_{3,1} \vee \neg v_{4,1} \vee \neg v_{5,1} \vee \neg v_{6,1})$$

$$\wedge (\neg v_{3,1} \vee \neg v_{4,2} \vee \neg v_{5,1} \vee \neg v_{6,1}) \wedge (\neg v_{3,1} \vee \neg v_{4,2} \vee \neg v_{5,1} \vee \neg v_{6,2}) \wedge (\neg v_{3,1} \vee \neg v_{4,2} \vee \neg v_{5,2} \vee \neg v_{6,1})$$

$$\wedge (\neg v_{3,2} \vee \neg v_{4,1} \vee \neg v_{5,1} \vee \neg v_{6,1}) \wedge (\neg v_{3,2} \vee \neg v_{4,1} \vee \neg v_{5,1} \vee \neg v_{6,2}) \wedge (\neg v_{3,2} \vee \neg v_{4,1} \vee \neg v_{5,2} \vee \neg v_{6,1})$$

$$\wedge (\neg v_{3,2} \vee \neg v_{4,2} \vee \neg v_{5,1} \vee \neg v_{6,1}) \wedge (\neg v_{3,2} \vee \neg v_{4,2} \vee \neg v_{5,1} \vee \neg v_{6,2}) \wedge (\neg v_{3,2} \vee \neg v_{4,2} \vee \neg v_{5,2} \vee \neg v_{6,1})$$

$$\wedge (\neg v_{3,2} \vee \neg v_{4,2} \vee \neg v_{5,2} \vee \neg v_{6,2}) \wedge (\neg v_{5,1} \vee \neg v_{6,1} \vee \neg v_{7,1} \vee \neg v_{8,1}) \wedge (\neg v_{5,1} \vee \neg v_{6,2} \vee \neg v_{7,1} \vee \neg v_{8,1})$$

$$\wedge (\neg v_{5,1} \vee \neg v_{6,2} \vee \neg v_{7,1} \vee \neg v_{8,2}) \wedge (\neg v_{5,1} \vee \neg v_{6,2} \vee \neg v_{7,2} \vee \neg v_{8,1}) \wedge (\neg v_{5,2} \vee \neg v_{6,1} \vee \neg v_{7,1} \vee \neg v_{8,1})$$

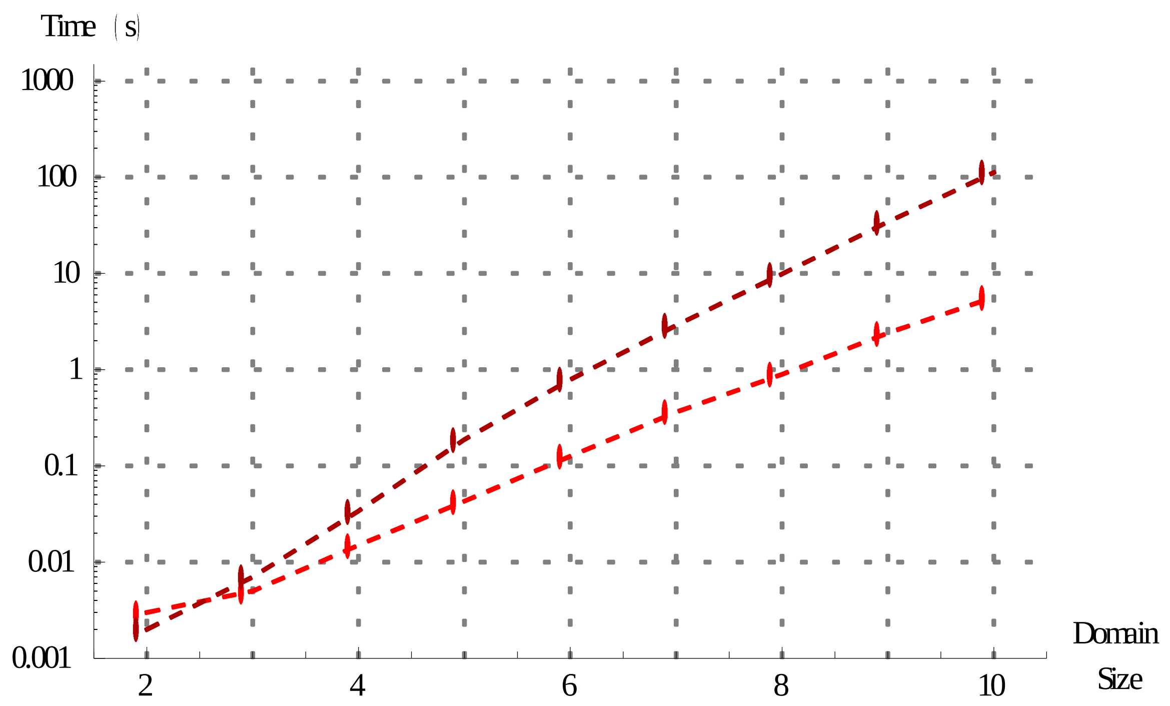
$$\wedge (\neg v_{5,2} \vee \neg v_{6,1} \vee \neg v_{7,1} \vee \neg v_{8,2}) \wedge (\neg v_{5,2} \vee \neg v_{6,1} \vee \neg v_{7,2} \vee \neg v_{8,1}) \wedge (\neg v_{5,2} \vee \neg v_{6,2} \vee \neg v_{7,1} \vee \neg v_{8,1})$$

$$\wedge (\neg v_{5,2} \vee \neg v_{6,2} \vee \neg v_{7,1} \vee \neg v_{8,2}) \wedge (\neg v_{5,2} \vee \neg v_{6,2} \vee \neg v_{7,2} \vee \neg v_{8,1}) \wedge (\neg v_{5,2} \vee \neg v_{6,2} \vee \neg v_{7,2} \vee \neg v_{8,2})$$

Direct Encoding

Solution by
SAT solver
MiniSAT

stripped down to a
simple SAT solver
with
clause-learning
random choice
always restart

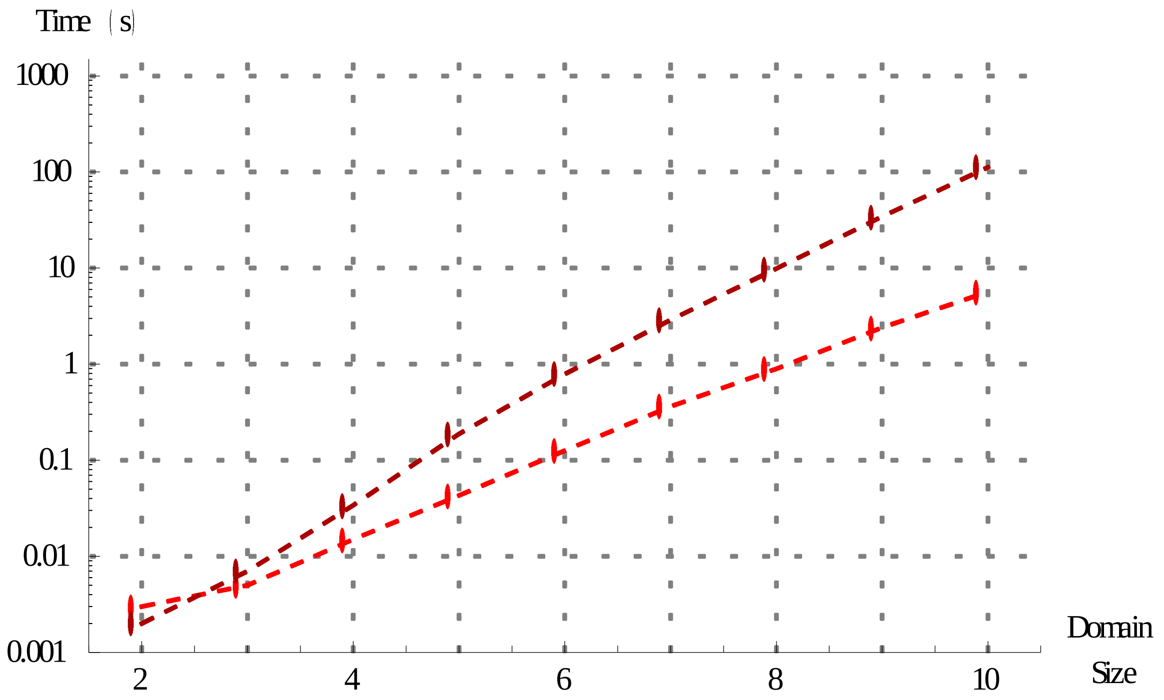


$$(v_{1,1} \lor v_{1,2}) \land (v_{2,1} \lor v_{2,2}) \land (v_{3,1} \lor v_{3,2}) \land (v_{4,1} \lor v_{4,2}) \land (v_{5,1} \lor v_{5,2}) \land (v_{6,1} \lor v_{6,2}) \land (v_{7,1} \lor v_{7,2}) \land (v_{8,1} \lor v_{8,2})$$
$$\land (\neg v_{1,1} \lor \neg v_{2,1} \lor \neg v_{3,1} \lor \neg v_{4,1}) \land (\neg v_{1,1} \lor \neg v_{2,2} \lor \neg v_{3,1} \lor \neg v_{4,1}) \land (\neg v_{1,1} \lor \neg v_{2,2} \lor \neg v_{3,1} \lor \neg v_{4,2})$$
$$\land (\neg v_{1,1} \lor \neg v_{2,2} \lor \neg v_{3,2} \lor \neg v_{4,1}) \land (\neg v_{1,2} \lor \neg v_{2,1} \lor \neg v_{3,1} \lor \neg v_{4,1}) \land (\neg v_{1,2} \lor \neg v_{2,1} \lor \neg v_{3,1} \lor \neg v_{4,2})$$
$$\land (\neg v_{1,2} \lor \neg v_{2,1} \lor \neg v_{3,2} \lor \neg v_{4,1}) \land (\neg v_{1,2} \lor \neg v_{2,2} \lor \neg v_{3,1} \lor \neg v_{4,1}) \land (\neg v_{1,2} \lor \neg v_{2,2} \lor \neg v_{3,1} \lor \neg v_{4,2})$$
$$\land (\neg v_{1,2} \lor \neg v_{2,2} \lor \neg v_{3,2} \lor \neg v_{4,1}) \land (\neg v_{1,2} \lor \neg v_{2,2} \lor \neg v_{3,2} \lor \neg v_{4,2}) \land (\neg v_{3,1} \lor \neg v_{4,1} \lor \neg v_{5,1} \lor \neg v_{6,1})$$
$$\land (\neg v_{3,1} \lor \neg v_{4,2} \lor \neg v_{5,1} \lor \neg v_{6,1}) \land (\neg v_{3,1} \lor \neg v_{4,2} \lor \neg v_{5,1} \lor \neg v_{6,2}) \land (\neg v_{3,1} \lor \neg v_{4,2} \lor \neg v_{5,2} \lor \neg v_{6,1})$$
$$\land (\neg v_{3,2} \lor \neg v_{4,1} \lor \neg v_{5,1} \lor \neg v_{6,1}) \land (\neg v_{3,2} \lor \neg v_{4,1} \lor \neg v_{5,1} \lor \neg v_{6,2}) \land (\neg v_{3,2} \lor \neg v_{4,1} \lor \neg v_{5,2} \lor \neg v_{6,1})$$
$$\land (\neg v_{3,2} \lor \neg v_{4,2} \lor \neg v_{5,1} \lor \neg v_{6,1}) \land (\neg v_{3,2} \lor \neg v_{4,2} \lor \neg v_{5,1} \lor \neg v_{6,2}) \land (\neg v_{3,2} \lor \neg v_{4,2} \lor \neg v_{5,2} \lor \neg v_{6,1})$$
$$\land (\neg v_{3,2} \lor \neg v_{4,2} \lor \neg v_{5,2} \lor \neg v_{6,2}) \land (\neg v_{5,1} \lor \neg v_{6,1} \lor \neg v_{7,1} \lor \neg v_{8,1}) \land (\neg v_{5,1} \lor \neg v_{6,2} \lor \neg v_{7,1} \lor \neg v_{8,1})$$
$$\land (\neg v_{5,1} \lor \neg v_{6,2} \lor \neg v_{7,1} \lor \neg v_{8,2}) \land (\neg v_{5,1} \lor \neg v_{6,2} \lor \neg v_{7,2} \lor \neg v_{8,1}) \land (\neg v_{5,2} \lor \neg v_{6,1} \lor \neg v_{7,1} \lor \neg v_{8,1})$$
$$\land (\neg v_{5,2} \lor \neg v_{6,1} \lor \neg v_{7,1} \lor \neg v_{8,2}) \land (\neg v_{5,2} \lor \neg v_{6,1} \lor \neg v_{7,2} \lor \neg v_{8,1}) \land (\neg v_{5,2} \lor \neg v_{6,2} \lor \neg v_{7,1} \lor \neg v_{8,1})$$
$$\land (\neg v_{5,2} \lor \neg v_{6,2} \lor \neg v_{7,1} \lor \neg v_{8,2}) \land (\neg v_{5,2} \lor \neg v_{6,2} \lor \neg v_{7,2} \lor \neg v_{8,1}) \land (\neg v_{5,2} \lor \neg v_{6,2} \lor \neg v_{7,2} \lor \neg v_{8,2})$$

Lesson II

Solution by
SAT solver
MiniSAT

stripped down to a
simple SAT solver
with
clause-learning
random choice
always restart

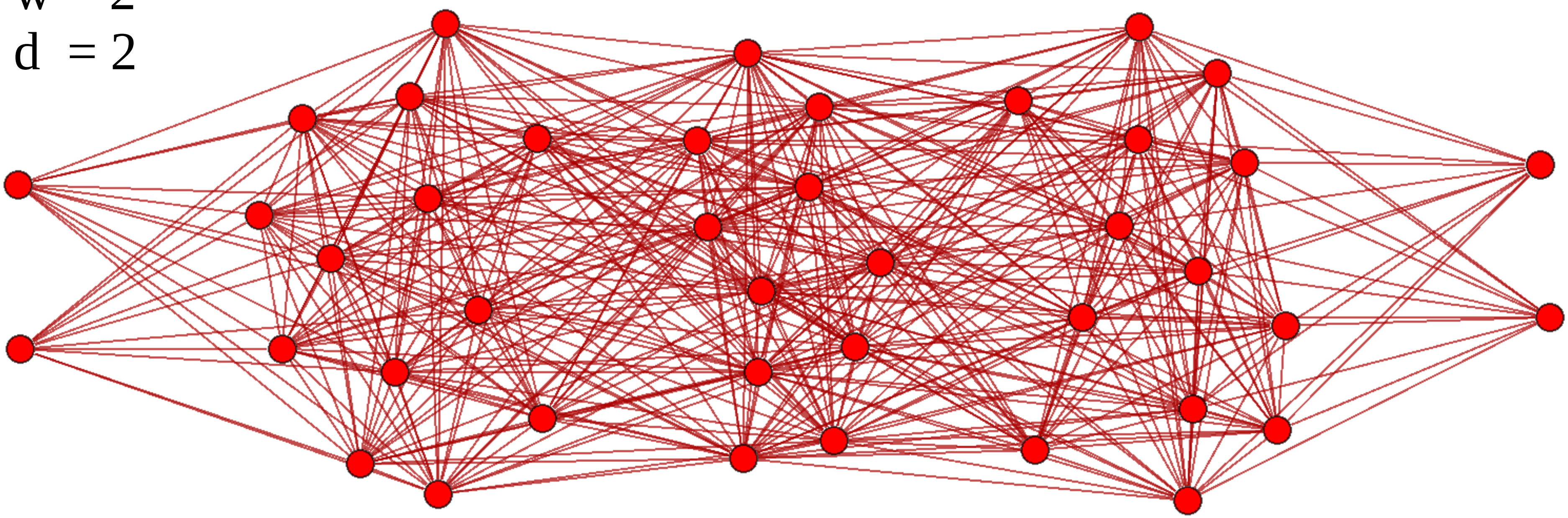


Hidden structure can be exploited by simple clause-learning SAT-solvers

WHERE’S THE STRUCTURE?

Structure in Direct Encoding

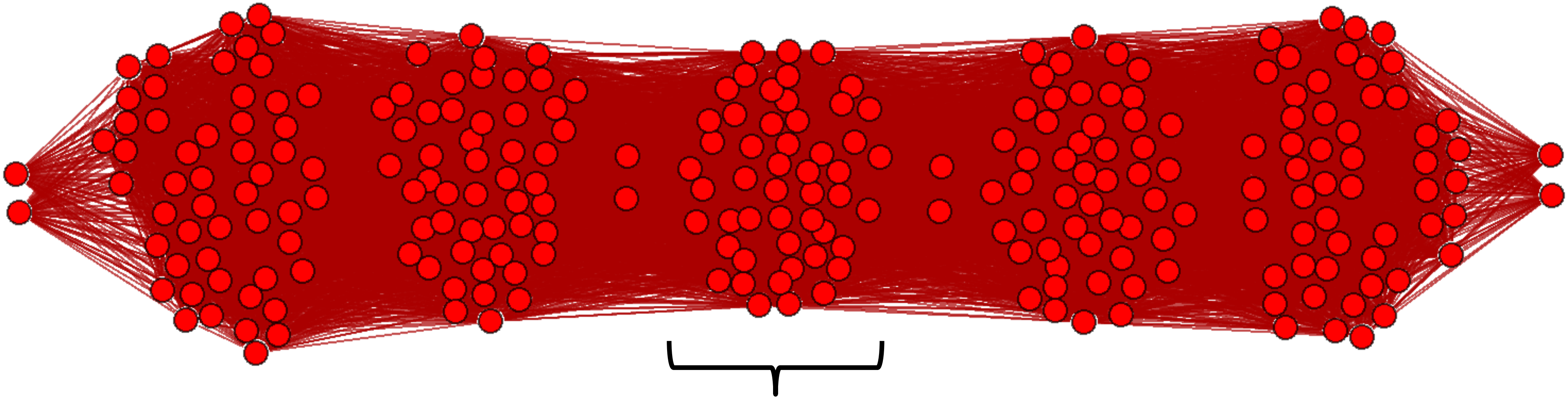
w = 2
d = 2



$$
\begin{aligned}
&(v_{1,1} \vee v_{1,2}) \wedge (v_{2,1} \vee v_{2,2}) \wedge (v_{3,1} \vee v_{3,2}) \wedge (v_{4,1} \vee v_{4,2}) \wedge (v_{5,1} \vee v_{5,2}) \wedge (v_{6,1} \vee v_{6,2}) \wedge (v_{7,1} \vee v_{7,2}) \wedge (v_{8,1} \vee v_{8,2}) \\
&\wedge (\neg v_{1,1} \vee \neg v_{2,1} \vee \neg v_{3,1} \vee \neg v_{4,1}) \wedge (\neg v_{1,1} \vee \neg v_{2,2} \vee \neg v_{3,1} \vee \neg v_{4,1}) \wedge (\neg v_{1,1} \vee \neg v_{2,2} \vee \neg v_{3,1} \vee \neg v_{4,2}) \\
&\wedge (\neg v_{1,1} \vee \neg v_{2,2} \vee \neg v_{3,2} \vee \neg v_{4,1}) \wedge (\neg v_{1,2} \vee \neg v_{2,1} \vee \neg v_{3,1} \vee \neg v_{4,1}) \wedge (\neg v_{1,2} \vee \neg v_{2,1} \vee \neg v_{3,1} \vee \neg v_{4,2}) \\
&\wedge (\neg v_{1,2} \vee \neg v_{2,1} \vee \neg v_{3,2} \vee \neg v_{4,1}) \wedge (\neg v_{1,2} \vee \neg v_{2,2} \vee \neg v_{3,1} \vee \neg v_{4,1}) \wedge (\neg v_{1,2} \vee \neg v_{2,2} \vee \neg v_{3,1} \vee \neg v_{4,2}) \\
&\wedge (\neg v_{1,2} \vee \neg v_{2,2} \vee \neg v_{3,2} \vee \neg v_{4,1}) \wedge (\neg v_{1,2} \vee \neg v_{2,2} \vee \neg v_{3,2} \vee \neg v_{4,2}) \wedge (\neg v_{3,1} \vee \neg v_{4,1} \vee \neg v_{5,1} \vee \neg v_{6,1}) \\
&\wedge (\neg v_{3,1} \vee \neg v_{4,2} \vee \neg v_{5,1} \vee \neg v_{6,1}) \wedge (\neg v_{3,1} \vee \neg v_{4,2} \vee \neg v_{5,1} \vee \neg v_{6,2}) \wedge (\neg v_{3,1} \vee \neg v_{4,2} \vee \neg v_{5,2} \vee \neg v_{6,1}) \\
&\wedge (\neg v_{3,2} \vee \neg v_{4,1} \vee \neg v_{5,1} \vee \neg v_{6,1}) \wedge (\neg v_{3,2} \vee \neg v_{4,1} \vee \neg v_{5,1} \vee \neg v_{6,2}) \wedge (\neg v_{3,2} \vee \neg v_{4,1} \vee \neg v_{5,2} \vee \neg v_{6,1}) \\
&\wedge (\neg v_{3,2} \vee \neg v_{4,2} \vee \neg v_{5,1} \vee \neg v_{6,1}) \wedge (\neg v_{3,2} \vee \neg v_{4,2} \vee \neg v_{5,1} \vee \neg v_{6,2}) \wedge (\neg v_{3,2} \vee \neg v_{4,2} \vee \neg v_{5,2} \vee \neg v_{6,1}) \\
&\wedge (\neg v_{3,2} \vee \neg v_{4,2} \vee \neg v_{5,2} \vee \neg v_{6,2}) \wedge (\neg v_{5,1} \vee \neg v_{6,1} \vee \neg v_{7,1} \vee \neg v_{8,1}) \wedge (\neg v_{5,1} \vee \neg v_{6,2} \vee \neg v_{7,1} \vee \neg v_{8,1}) \\
&\wedge (\neg v_{5,1} \vee \neg v_{6,2} \vee \neg v_{7,1} \vee \neg v_{8,2}) \wedge (\neg v_{5,1} \vee \neg v_{6,2} \vee \neg v_{7,2} \vee \neg v_{8,1}) \wedge (\neg v_{5,2} \vee \neg v_{6,1} \vee \neg v_{7,1} \vee \neg v_{8,1}) \\
&\wedge (\neg v_{5,2} \vee \neg v_{6,1} \vee \neg v_{7,1} \vee \neg v_{8,2}) \wedge (\neg v_{5,2} \vee \neg v_{6,1} \vee \neg v_{7,2} \vee \neg v_{8,1}) \wedge (\neg v_{5,2} \vee \neg v_{6,2} \vee \neg v_{7,1} \vee \neg v_{8,1}) \\
&\wedge (\neg v_{5,2} \vee \neg v_{6,2} \vee \neg v_{7,1} \vee \neg v_{8,2}) \wedge (\neg v_{5,2} \vee \neg v_{6,2} \vee \neg v_{7,2} \vee \neg v_{8,1}) \wedge (\neg v_{5,2} \vee \neg v_{6,2} \vee \neg v_{7,2} \vee \neg v_{8,2})
\end{aligned}
$$

Structure in Direct Encoding

w = 2
d = 3



$O(d^{2w})$ clauses

2wd variables

Tree-width is 2wd-1

Structure in Direct Encoding

$$\neg \mathbf{v}_{1,1} \vee \neg \mathbf{v}_{2,1} \vee \neg \mathbf{v}_{3,1} \vee \neg \mathbf{v}_{4,1}$$

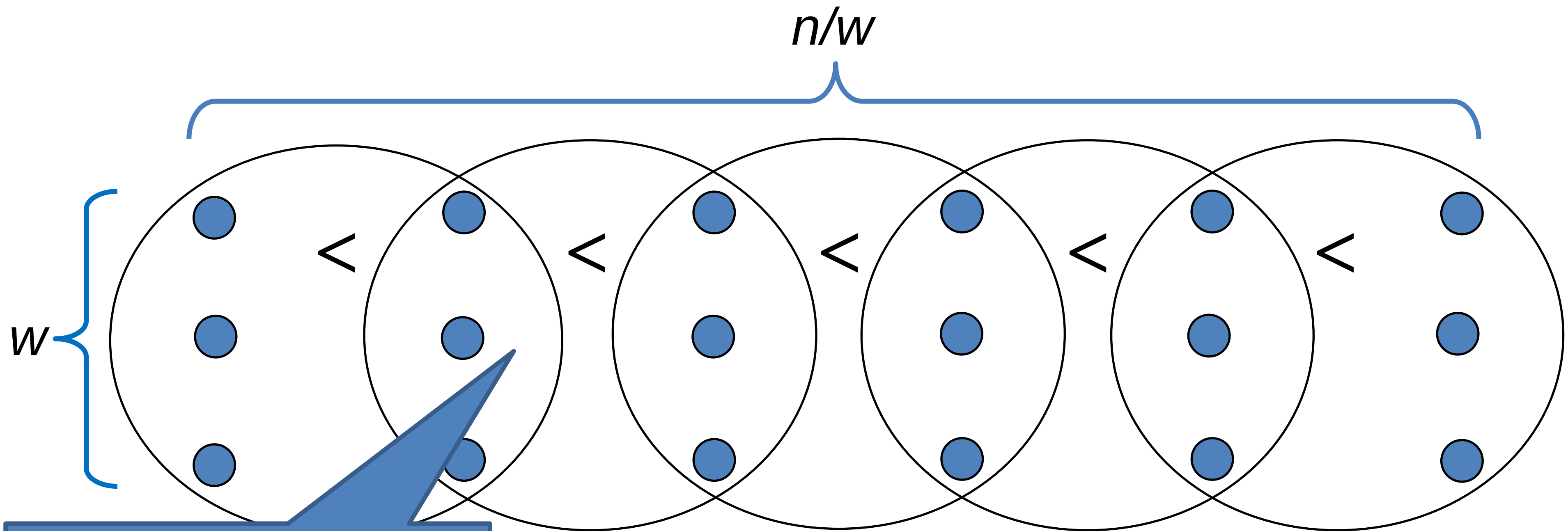
$$\mathbf{v}_{1,1} \vee \mathbf{v}_{1,2} \vee \ldots \vee \mathbf{v}_{1,d}$$

$$\neg \mathbf{v}_{2,1} \vee \neg \mathbf{v}_{3,1} \vee \neg \mathbf{v}_{4,1} \vee \mathbf{v}_{1,2} \vee \ldots \vee \mathbf{v}_{1,d}$$

$2w - 2 + \mathbf{d}$

Width of resolution proof is at least 2w-2+d

Structure in the CSP Model



Domain is {1,2,...,d}

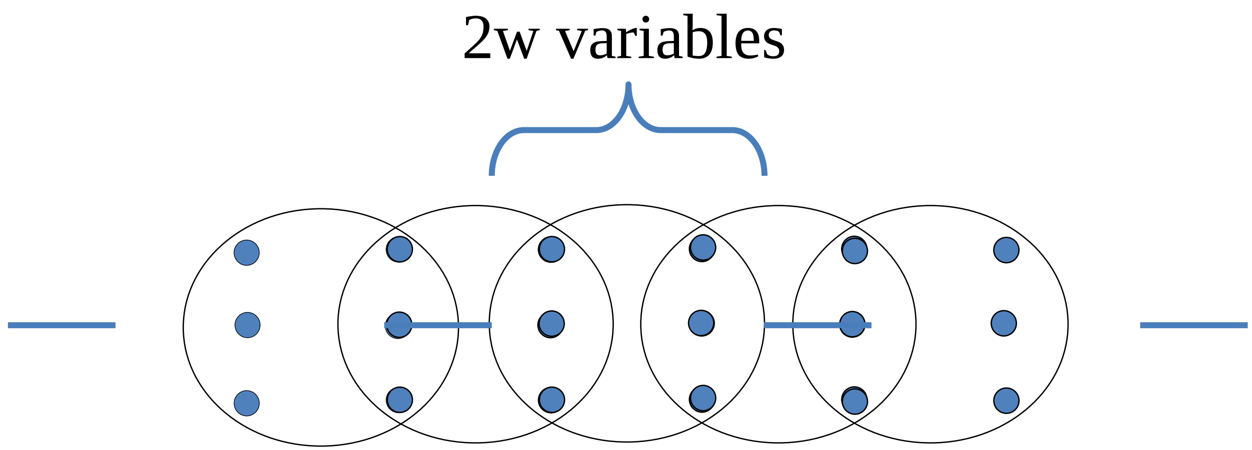
No value can be eliminated

(This is why CSP solvers have trouble)

```
array[1..2d] of var 1..d:X1;
array[1..2d] of var 1..d:X2;
constraint
forall(i in 1..2d-1)(
    X1[i]+X2[i] < X1[i+1]+X2[i+1]);
solve satisfy;
```

Specification in MiniZinc for *w=2*

Structure in the CSP Model



Tree-width is 2w-1

Independent of d

in polynomial time (for fixed w)

Can be solved by establishing (2w)-consistency...

(Atserias, Bulatov & Dalmau, 2007)

COMPUTING CONSISTENCY

Local Consistency

Local consistency is a key concept for the CSP, going back to Montanari's original paper in 1974, and Freuder in 1978.

A CSP is said to be k-consistent, if, for all sets of k variables, $\{v_1,v_2,...,v_k\}$

the assignment $v_1=a_1,v_2=a_1,....v_{k-1}=a_{k-1}$
is allowed by all constraints

$\Downarrow$

there exists some value a_k such that
the assignment $v_1=a_1,v_2=a_2,....v_{k-1}=a_{k-1},v_k=a_k$
is allowed by all constraints

"The best framework for understanding all the network consistency algorithms is to see them as ***removing local inconsistencies*** *from the network which can never be part of any global solution."* (Freuder 2006)

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$\Downarrow$

there exists some value a_k such that
the assignment $v_1=a_1,v_2=a_2,....v_{k-1}=a_{k-1},v_k=a_k$
is allowed by all constraints

for all possible values a_k
the assignment $v_1=a_1,v_2=a_1,....v_{k-1}=a_{k-1},v_k=a_k$
is disallowed by some constraint

$\Downarrow$

the assignment $v_1=a_1,v_2=a_2,....v_{k-1}=a_{k-1}$
is disallowed by some constraint

"The best framework for understanding all the network consistency algorithms is to see them as ***removing local inconsistencies*** *from the network which can never be part of any global solution."* (Freuder 2006)

Local Consistency and SAT

Local consistency is a key concept for the CSP, going back to Montanari's original paper in 1974, and Freuder in 1978.

A CSP is said to be k-consistent, if, for all sets of k variables, $\{v_1,v_2,\ldots,v_k\}$

$$v_{1,1} \lor v_{1,2} \lor \ldots \lor v_{1,d}$$

$$\neg v_{1,1} \lor \neg v_{2,1} \lor \neg v_{3,1} \lor \neg v_{4,1}$$

$$\neg v_{1,2} \lor \neg v_{2,1} \lor \neg v_{3,1} \lor \neg v_{4,1}$$

$$\vdots$$

$$\neg v_{1,d} \lor \neg v_{2,1} \lor \neg v_{3,1} \lor \neg v_{4,1}$$

$$\neg v_{2,1} \lor \neg v_{3,1} \lor \neg v_{4,1}$$

for all possible values a_k
the assignment $v_1=a_1,v_2=a_1,\ldots.v_{k-1}=a_{k-1},v_k=a_k$
is disallowed by some constraint

$$\Downarrow$$

the assignment $v_1=a_1,v_2=a_2,\ldots.v_{k-1}=a_{k-1}$
is disallowed by some constraint

"The best framework for understanding all the network consistency algorithms is to see them as ***removing local inconsistencies*** *from the network which can never be part of any global solution."* (Freuder 2006)

Hyper-resolution

This extended form of resolution has been called:

- the "nogood resolution rule" Hwang & Mitchell (2005)
- the "nogood recording scheme" Schiex & Verfaillie (1993)
- the "H5-k rule" de Kleer (1989)

$$v_{1,1} \vee v_{1,2} \vee \ldots \vee v_{1,d}$$
$$\neg v_{1,1} \vee \neg v_{2,1} \vee \neg v_{3,1} \vee \neg v_{4,1}$$
$$\neg v_{1,2} \vee \neg v_{2,1} \vee \neg v_{3,1} \vee \neg v_{4,1}$$
$$\vdots$$
$$\neg v_{1,d} \vee \neg v_{2,1} \vee \neg v_{3,1} \vee \neg v_{4,1}$$
$$\overline{\neg v_{2,1} \vee \neg v_{3,1} \vee \neg v_{4,1}}$$

$$x_1 \vee x_2 \vee \ldots \vee x_r \vee C_0$$
$$\neg x_1 \vee C_1$$
$$\neg x_2 \vee C_2$$
$$\vdots$$
$$\neg x_r \vee C_r$$
$$\overline{C_0 \vee C_1 \vee C_2 \vee \ldots \vee C_r}$$

Theorem: The k-consistency closure of a CSP instance is empty *if and only if* its direct encoding has a hyper-resolution refutation of width $\leq$ k.

Jeavons & Petke (2012)

Hyper-resolution and SAT-solvers

Theorem: The k-consistency closure of a CSP instance is empty *if and only if* its direct encoding has a hyper-resolution refutation of width $\leq$ k.

Theorem: If a set of non-empty clauses over n Boolean variables has a hyper-resolution refutation of width k and length m, then the ***expected number of restarts*** required by a standard randomised SAT-solver using the Decision learning scheme to discover that they are unsatisfiable is $< m \binom{n}{k}$.

$$\frac{\begin{array}{l} x_1 \vee x_2 \vee \ldots \vee x_r \vee C_0 \\ \neg x_1 \vee C_1 \\ \neg x_2 \vee C_2 \\ \vdots \\ \neg x_r \vee C_r \end{array}}{C_0 \vee C_1 \vee C_2 \vee \ldots \vee C_r}$$

Theorem: If a CSP instance can be solved by establishing k-consistency, then the ***expected number of restarts*** required by a standard randomised SAT-solver on its direct encoding is $O(n^{2k}d^{2k})$.

Hyper-resolution and SAT-solvers

"Local Consistency and SAT-Solvers"
Peter Jeavons & Justyna Petke
Journal of Artificial Intelligence Research 43 (2012)

Summary: SAT-solvers emulate k-consistency in expected polynomial time

clause-learning

for any fixed k

Lesson III

"Local Consistency and SAT-Solvers"
Peter Jeavons & Justyna Petke
Journal of Artificial Intelligence Research 43 (2012)

Summary: SAT-solvers emulate k-consistency in expected polynomial time

clause-learning

on the direct encoding

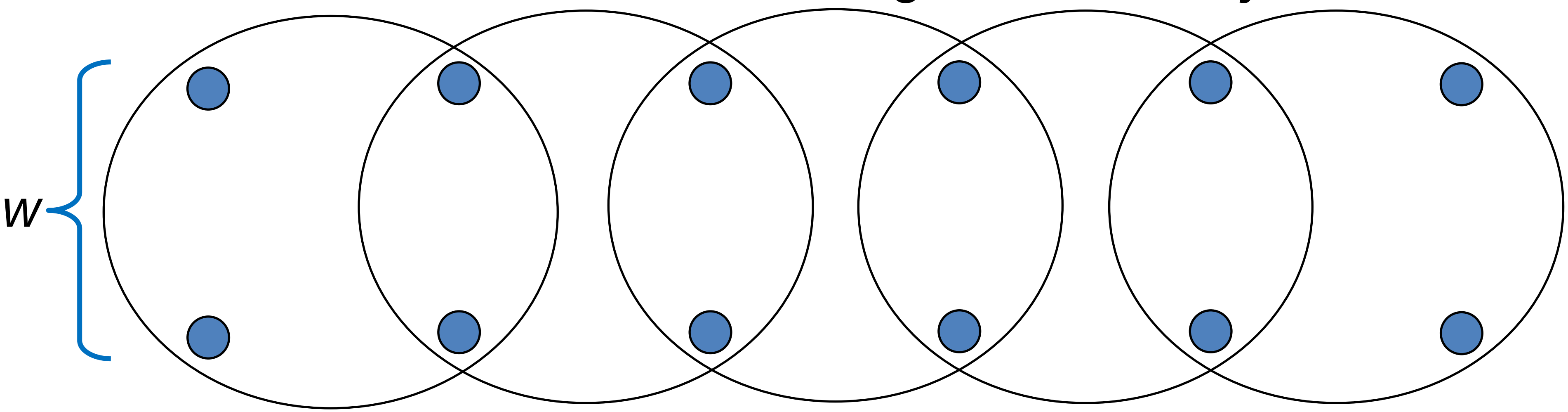
for any fixed k

Learning is a powerful and flexible mechanism to exploit hidden structure

OTHER ENCODINGS?

A Family of Examples

$a + b < c + d$ $\quad$ $e + f < g + h$ $\quad$ $i + j < k + l$



$c + d < e + f$ $\quad$ $g + h < i + j$

Order Encoding

Variables: $x_{i,\leq 1}, x_{i,\leq 2}, x_{i,\leq 3}, \ldots, x_{i,\leq d-1}$

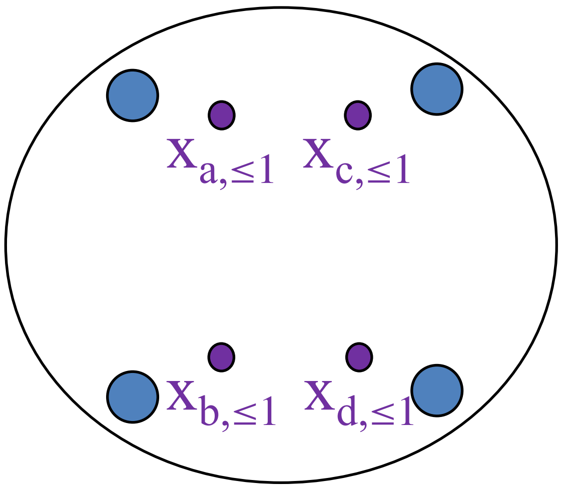
Consistency clauses:

$$(\neg x_{i,\leq 1} \vee x_{i,\leq 2}) \wedge (\neg x_{i,\leq 2} \vee x_{i,\leq 3}) \wedge \ldots \wedge (\neg x_{i,\leq d-2} \vee x_{i,\leq d-1})$$

All 2-clauses
(for any d)

Order Encoding

a + b < c + d



domain: *{1,2}*

a	b	c	d	
1	1	1	1	✗
1	1	1	2	✓
1	1	2	1	✓
1	1	2	2	✓
1	2	1	1	✗
1	2	1	2	✗
1	2	2	1	✗
1	2	2	2	✓
2	1	1	1	✗
2	1	1	2	✗
2	1	2	1	✗
2	1	2	2	✓
2	2	1	1	✗
2	2	1	2	✗
2	2	2	1	✗
2	2	2	2	✗

$$(\neg x_{c,\leq 1} \vee \neg x_{d,\leq 1})$$

$$\wedge$$

$$(x_{b,\leq 1} \vee \neg x_{d,\leq 1})$$

$$\wedge$$

$$(x_{b,\leq 1} \vee \neg x_{c,\leq 1})$$

$$\wedge$$

$$(x_{a,\leq 1} \vee \neg x_{d,\leq 1})$$

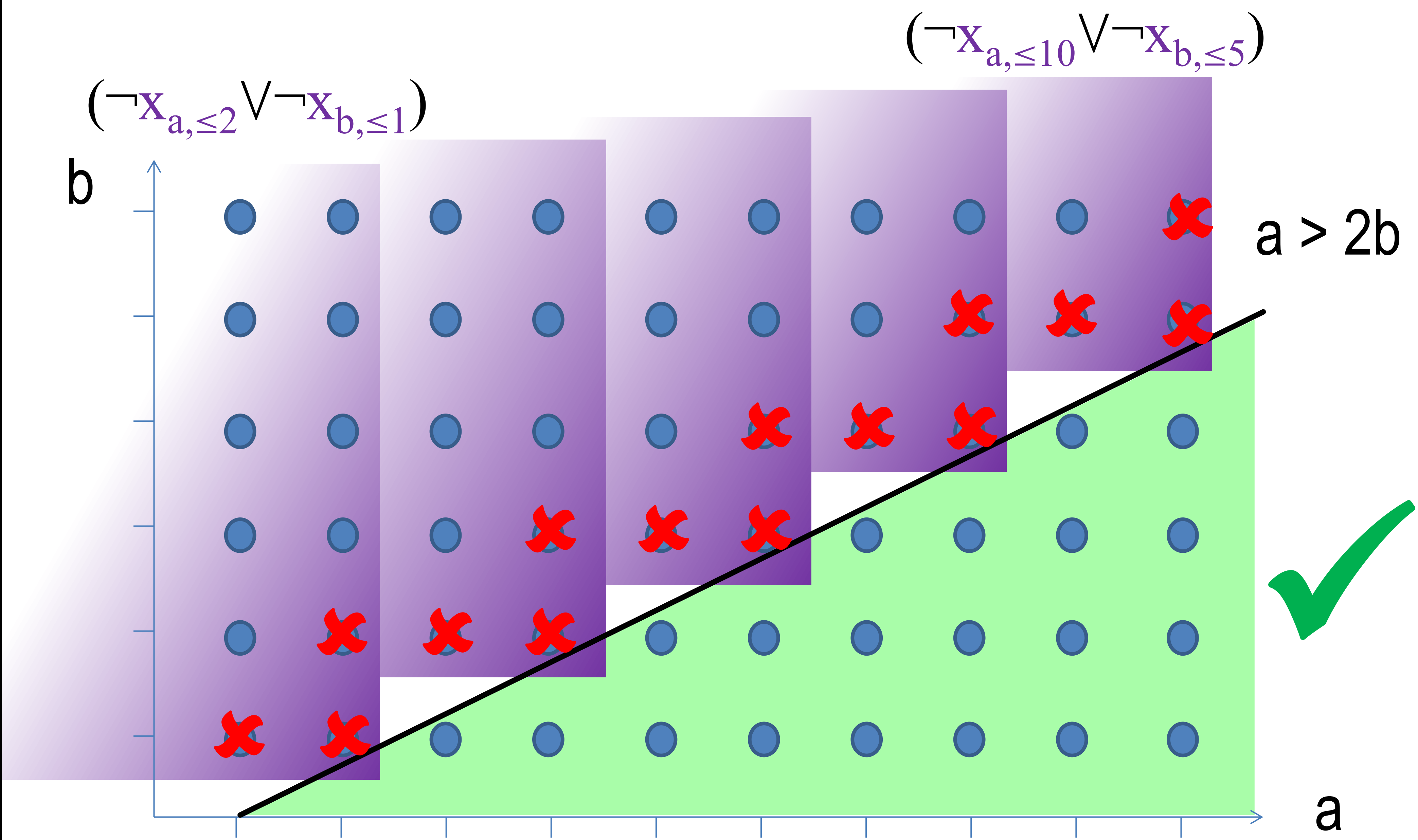
$$\wedge$$

$$(x_{a,\leq 1} \vee \neg x_{c,\leq 1})$$

$$\wedge$$

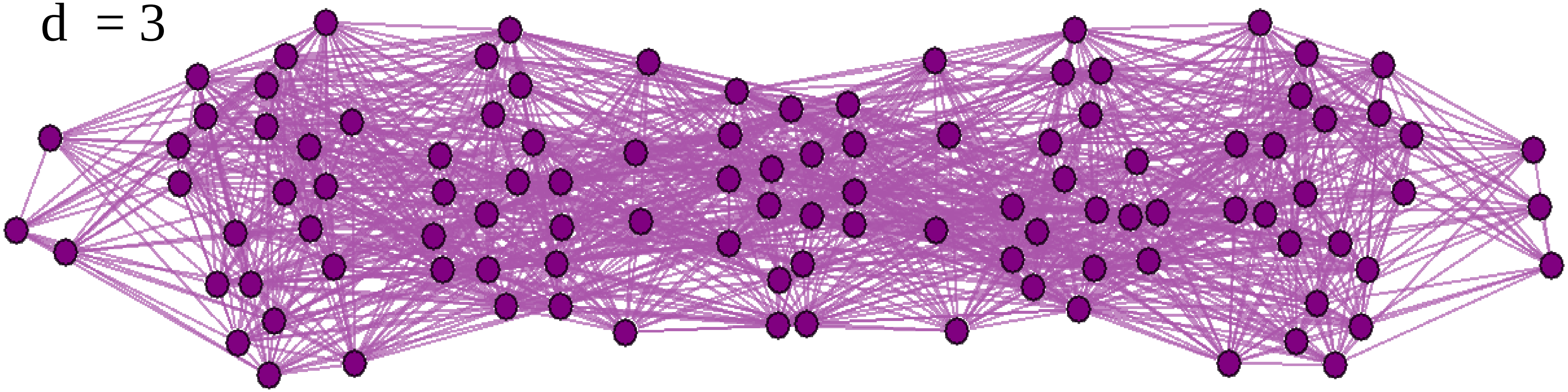
$$(x_{a,\leq 1} \vee x_{b,\leq 1})$$

Order Encoding



Order Encoding

w = 2
d = 3



$(\neg x_{1,\le1} \vee x_{1,\le2}) \wedge (\neg x_{2,\le1} \vee x_{2,\le2}) \wedge (\neg x_{3,\le1} \vee x_{3,\le2}) \wedge (\neg x_{4,\le1} \vee x_{4,\le2}) \wedge (\neg x_{5,\le1} \vee x_{5,\le2}) \wedge (\neg x_{6,\le1} \vee x_{6,\le2}) \wedge$
$(\neg x_{7,\le1} \vee x_{7,\le2}) \wedge (\neg x_{8,\le1} \vee x_{8,\le2}) \wedge (\neg x_{9,\le1} \vee x_{9,\le2}) \wedge (\neg x_{10,\le1} \vee x_{10,\le2}) \wedge (\neg x_{11,\le1} \vee x_{11,\le2}) \wedge (\neg x_{12,\le1} \vee x_{12,\le2}) \wedge$

$(\neg x_{3,\le1} \vee \neg x_{4,\le1}) \wedge (x_{2,\le1} \vee \neg x_{3,\le2} \vee \neg x_{4,\le1}) \wedge (x_{2,\le1} \vee \neg x_{3,\le1} \vee \neg x_{4,\le2}) \wedge (x_{2,\le2} \vee \neg x_{4,\le1}) \wedge (x_{2,\le2} \vee \neg x_{3,\le2} \vee \neg x_{4,\le2}) \wedge (x_{2,\le2} \vee \neg x_{3,\le1}) \wedge (x_{1,\le1} \vee \neg x_{3,\le2} \vee \neg x_{4,\le1}) \wedge$
$(x_{1,\le1} \vee \neg x_{3,\le1} \vee \neg x_{4,\le2}) \wedge (x_{1,\le1} \vee x_{2,\le1} \vee \neg x_{4,\le1}) \wedge (x_{1,\le1} \vee x_{2,\le1} \vee \neg x_{3,\le2} \vee \neg x_{4,\le2}) \wedge (x_{1,\le1} \vee x_{2,\le1} \vee \neg x_{3,\le1}) \wedge (x_{1,\le1} \vee x_{2,\le2} \vee \neg x_{4,\le2}) \wedge (x_{1,\le1} \vee x_{2,\le2} \vee \neg x_{3,\le2}) \wedge (x_{1,\le2} \vee \neg x_{4,\le1}) \wedge$
$(x_{1,\le2} \vee \neg x_{3,\le2} \vee \neg x_{4,\le2}) \wedge (x_{1,\le2} \vee \neg x_{3,\le1}) \wedge (x_{1,\le2} \vee x_{2,\le1} \vee \neg x_{4,\le2}) \wedge (x_{1,\le2} \vee x_{2,\le1} \vee \neg x_{3,\le2}) \wedge (x_{1,\le2} \vee x_{2,\le2}) \wedge$

$(\neg x_{5,\le1} \vee \neg x_{6,\le1}) \wedge (x_{4,\le1} \vee \neg x_{5,\le2} \vee \neg x_{6,\le1}) \wedge (x_{4,\le1} \vee \neg x_{5,\le1} \vee \neg x_{6,\le2}) \wedge (x_{4,\le2} \vee \neg x_{6,\le1}) \wedge (x_{4,\le2} \vee \neg x_{5,\le2} \vee \neg x_{6,\le2}) \wedge (x_{4,\le2} \vee \neg x_{5,\le1}) \wedge (x_{3,\le1} \vee \neg x_{5,\le2} \vee \neg x_{6,\le1}) \wedge$
$(x_{3,\le1} \vee \neg x_{5,\le1} \vee \neg x_{6,\le2}) \wedge (x_{3,\le1} \vee x_{4,\le1} \vee \neg x_{6,\le1}) \wedge (x_{3,\le1} \vee x_{4,\le1} \vee \neg x_{5,\le2} \vee \neg x_{6,\le2}) \wedge (x_{3,\le1} \vee x_{4,\le1} \vee \neg x_{5,\le1}) \wedge (x_{3,\le1} \vee x_{4,\le2} \vee \neg x_{6,\le2}) \wedge (x_{3,\le1} \vee x_{4,\le2} \vee \neg x_{5,\le2}) \wedge (x_{3,\le2} \vee \neg x_{6,\le1}) \wedge$
$(x_{3,\le2} \vee \neg x_{5,\le2} \vee \neg x_{6,\le2}) \wedge (x_{3,\le2} \vee \neg x_{5,\le1}) \wedge (x_{3,\le2} \vee x_{4,\le1} \vee \neg x_{6,\le2}) \wedge (x_{3,\le2} \vee x_{4,\le1} \vee \neg x_{5,\le2}) \wedge (x_{3,\le2} \vee x_{4,\le2}) \wedge$

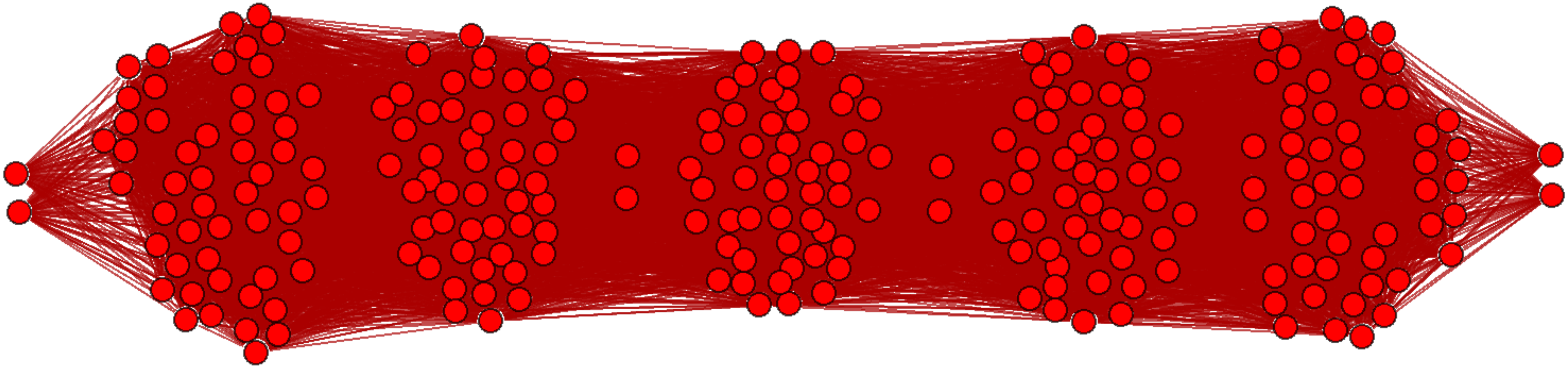
$(\neg x_{7,\le1} \vee \neg x_{8,\le1}) \wedge (x_{6,\le1} \vee \neg x_{7,\le2} \vee \neg x_{8,\le1}) \wedge (x_{6,\le1} \vee \neg x_{7,\le1} \vee \neg x_{8,\le2}) \wedge (x_{6,\le2} \vee \neg x_{8,\le1}) \wedge (x_{6,\le2} \vee \neg x_{7,\le2} \vee \neg x_{8,\le2}) \wedge (x_{6,\le2} \vee \neg x_{7,\le1}) \wedge (x_{5,\le1} \vee \neg x_{7,\le2} \vee \neg x_{8,\le1}) \wedge$
$(x_{5,\le1} \vee \neg x_{7,\le1} \vee \neg x_{8,\le2}) \wedge (x_{5,\le1} \vee x_{6,\le1} \vee \neg x_{8,\le1}) \wedge (x_{5,\le1} \vee x_{6,\le1} \vee \neg x_{7,\le2} \vee \neg x_{8,\le2}) \wedge (x_{5,\le1} \vee x_{6,\le1} \vee \neg x_{7,\le1}) \wedge (x_{5,\le1} \vee x_{6,\le2} \vee \neg x_{8,\le2}) \wedge (x_{5,\le1} \vee x_{6,\le2} \vee \neg x_{7,\le2}) \wedge (x_{5,\le2} \vee \neg x_{8,\le1}) \wedge$
$(x_{5,\le2} \vee \neg x_{7,\le2} \vee \neg x_{8,\le2}) \wedge (x_{5,\le2} \vee \neg x_{7,\le1}) \wedge (x_{5,\le2} \vee x_{6,\le1} \vee \neg x_{8,\le2}) \wedge (x_{5,\le2} \vee x_{6,\le1} \vee \neg x_{7,\le2}) \wedge (x_{5,\le2} \vee x_{6,\le2}) \wedge$

$(\neg x_{9,\le1} \vee \neg x_{10,\le1}) \wedge (x_{8,\le1} \vee \neg x_{9,\le2} \vee \neg x_{10,\le1}) \wedge (x_{8,\le1} \vee \neg x_{9,\le1} \vee \neg x_{10,\le2}) \wedge (x_{8,\le2} \vee \neg x_{10,\le1}) \wedge (x_{8,\le2} \vee \neg x_{9,\le2} \vee \neg x_{10,\le2}) \wedge (x_{8,\le2} \vee \neg x_{9,\le1}) \wedge (x_{7,\le1} \vee \neg x_{9,\le2} \vee \neg x_{10,\le1}) \wedge$
$(x_{7,\le1} \vee \neg x_{9,\le1} \vee \neg x_{10,\le2}) \wedge (x_{7,\le1} \vee x_{8,\le1} \vee \neg x_{10,\le1}) \wedge (x_{7,\le1} \vee x_{8,\le1} \vee \neg x_{9,\le2} \vee \neg x_{10,\le2}) \wedge (x_{7,\le1} \vee x_{8,\le1} \vee \neg x_{9,\le1}) \wedge (x_{7,\le1} \vee x_{8,\le2} \vee \neg x_{10,\le2}) \wedge (x_{7,\le1} \vee x_{8,\le2} \vee \neg x_{9,\le2}) \wedge$
$(x_{7,\le2} \vee \neg x_{10,\le1}) \wedge (x_{7,\le2} \vee \neg x_{9,\le2} \vee \neg x_{10,\le2}) \wedge (x_{7,\le2} \vee \neg x_{9,\le1}) \wedge (x_{7,\le2} \vee x_{8,\le1} \vee \neg x_{10,\le2}) \wedge (x_{7,\le2} \vee x_{8,\le1} \vee \neg x_{9,\le2}) \wedge (x_{7,\le2} \vee x_{8,\le2}) \wedge$

$(\neg x_{11,\le1} \vee \neg x_{12,\le1}) \wedge (x_{10,\le1} \vee \neg x_{11,\le2} \vee \neg x_{12,\le1}) \wedge (x_{10,\le1} \vee \neg x_{11,\le1} \vee \neg x_{12,\le2}) \wedge (x_{10,\le2} \vee \neg x_{12,\le1}) \wedge (x_{10,\le2} \vee \neg x_{11,\le2} \vee \neg x_{12,\le2}) \wedge (x_{10,\le2} \vee \neg x_{11,\le1}) \wedge (x_{9,\le1} \vee \neg x_{11,\le2} \vee \neg x_{12,\le1})$
$\wedge (x_{9,\le1} \vee \neg x_{11,\le1} \vee \neg x_{12,\le2}) \wedge (x_{9,\le1} \vee x_{10,\le1} \vee \neg x_{12,\le1}) \wedge (x_{9,\le1} \vee x_{10,\le1} \vee \neg x_{11,\le2} \vee \neg x_{12,\le2}) \wedge (x_{9,\le1} \vee x_{10,\le1} \vee \neg x_{11,\le1}) \wedge (x_{9,\le1} \vee x_{10,\le2} \vee \neg x_{12,\le2}) \wedge (x_{9,\le1} \vee x_{10,\le2} \vee \neg x_{11,\le2}) \wedge$
$(x_{9,\le2} \vee \neg x_{12,\le1}) \wedge (x_{9,\le2} \vee \neg x_{11,\le2} \vee \neg x_{12,\le2}) \wedge (x_{9,\le2} \vee \neg x_{11,\le1}) \wedge (x_{9,\le2} \vee x_{10,\le1} \vee \neg x_{12,\le2}) \wedge (x_{9,\le2} \vee x_{10,\le1} \vee \neg x_{11,\le2}) \wedge (x_{9,\le2} \vee x_{10,\le2})$

Direct Encoding

$w = 2$
$d = 3$



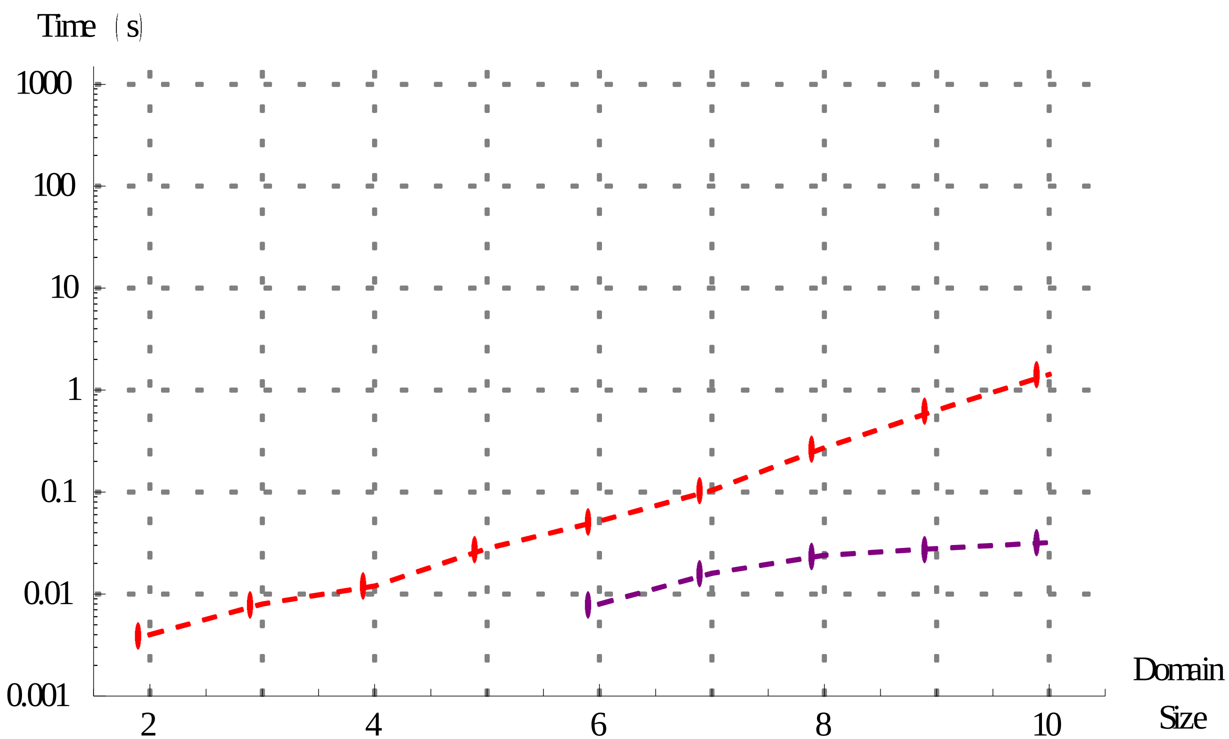
$(v_{1,1}\vee v_{1,2}\vee v_{1,3}) \wedge (v_{2,1}\vee v_{2,2}\vee v_{2,3}) \wedge (v_{3,1}\vee v_{3,2}\vee v_{3,3}) \wedge (v_{4,1}\vee v_{4,2}\vee v_{4,3}) \wedge (v_{5,1}\vee v_{5,2}\vee v_{5,3}) \wedge (v_{6,1}\vee v_{6,2}\vee v_{6,3}) \wedge$
$(v_{7,1}\vee v_{7,2}\vee v_{7,3}) \wedge (v_{8,1}\vee v_{8,2}\vee v_{8,3}) \wedge (v_{9,1}\vee v_{9,2}\vee v_{9,3}) \wedge (v_{10,1}\vee v_{10,2}\vee v_{10,3}) \wedge (v_{11,1}\vee v_{11,2}\vee v_{11,3}) \wedge (v_{12,1}\vee v_{12,2}\vee v_{12,3}) \wedge$

$(\neg v_{1,1}\vee\neg v_{2,1}\vee\neg v_{3,1}\vee\neg v_{4,1}) \wedge (\neg v_{1,1}\vee\neg v_{2,2}\vee\neg v_{3,1}\vee\neg v_{4,1}) \wedge (\neg v_{1,1}\vee\neg v_{2,2}\vee\neg v_{3,1}\vee\neg v_{4,2}) \wedge (\neg v_{1,1}\vee\neg v_{2,2}\vee\neg v_{3,2}\vee\neg v_{4,1}) \wedge$
$(\neg v_{1,1}\vee\neg v_{2,3}\vee\neg v_{3,1}\vee\neg v_{4,1}) \wedge (\neg v_{1,1}\vee\neg v_{2,3}\vee\neg v_{3,1}\vee\neg v_{4,2}) \wedge (\neg v_{1,1}\vee\neg v_{2,3}\vee\neg v_{3,1}\vee\neg v_{4,3}) \wedge (\neg v_{1,1}\vee\neg v_{2,3}\vee\neg v_{3,2}\vee\neg v_{4,1}) \wedge$
$(\neg v_{1,1}\vee\neg v_{2,3}\vee\neg v_{3,2}\vee\neg v_{4,2}) \wedge (\neg v_{1,1}\vee\neg v_{2,3}\vee\neg v_{3,3}\vee\neg v_{4,1}) \wedge (\neg v_{1,2}\vee\neg v_{2,1}\vee\neg v_{3,1}\vee\neg v_{4,1}) \wedge (\neg v_{1,2}\vee\neg v_{2,1}\vee\neg v_{3,1}\vee\neg v_{4,2}) \wedge$
$(\neg v_{1,2}\vee\neg v_{2,1}\vee\neg v_{3,2}\vee\neg v_{4,1}) \wedge (\neg v_{1,2}\vee\neg v_{2,2}\vee\neg v_{3,1}\vee\neg v_{4,1}) \wedge (\neg v_{1,2}\vee\neg v_{2,2}\vee\neg v_{3,1}\vee\neg v_{4,2}) \wedge (\neg v_{1,2}\vee\neg v_{2,2}\vee\neg v_{3,1}\vee\neg v_{4,3}) \wedge$
$(\neg v_{1,2}\vee\neg v_{2,2}\vee\neg v_{3,2}\vee\neg v_{4,1}) \wedge (\neg v_{1,2}\vee\neg v_{2,2}\vee\neg v_{3,2}\vee\neg v_{4,2}) \wedge (\neg v_{1,2}\vee\neg v_{2,2}\vee\neg v_{3,3}\vee\neg v_{4,1}) \wedge (\neg v_{1,2}\vee\neg v_{2,3}\vee\neg v_{3,1}\vee\neg v_{4,1}) \wedge$
$(\neg v_{1,2}\vee\neg v_{2,3}\vee\neg v_{3,1}\vee\neg v_{4,2}) \wedge (\neg v_{1,2}\vee\neg v_{2,3}\vee\neg v_{3,1}\vee\neg v_{4,3}) \wedge (\neg v_{1,2}\vee\neg v_{2,3}\vee\neg v_{3,2}\vee\neg v_{4,1}) \wedge (\neg v_{1,2}\vee\neg v_{2,3}\vee\neg v_{3,2}\vee\neg v_{4,2}) \wedge$
$(\neg v_{1,2}\vee\neg v_{2,3}\vee\neg v_{3,2}\vee\neg v_{4,3}) \wedge (\neg v_{1,2}\vee\neg v_{2,3}\vee\neg v_{3,3}\vee\neg v_{4,1}) \wedge (\neg v_{1,2}\vee\neg v_{2,3}\vee\neg v_{3,3}\vee\neg v_{4,2}) \wedge (\neg v_{1,3}\vee\neg v_{2,1}\vee\neg v_{3,1}\vee\neg v_{4,1}) \wedge$
$(\neg v_{1,3}\vee\neg v_{2,1}\vee\neg v_{3,1}\vee\neg v_{4,2}) \wedge (\neg v_{1,3}\vee\neg v_{2,1}\vee\neg v_{3,1}\vee\neg v_{4,3}) \wedge (\neg v_{1,3}\vee\neg v_{2,1}\vee\neg v_{3,2}\vee\neg v_{4,1}) \wedge (\neg v_{1,3}\vee\neg v_{2,1}\vee\neg v_{3,2}\vee\neg v_{4,2}) \wedge$
$(\neg v_{1,3}\vee\neg v_{2,1}\vee\neg v_{3,3}\vee\neg v_{4,1}) \wedge (\neg v_{1,3}\vee\neg v_{2,2}\vee\neg v_{3,1}\vee\neg v_{4,1}) \wedge (\neg v_{1,3}\vee\neg v_{2,2}\vee\neg v_{3,1}\vee\neg v_{4,2}) \wedge (\neg v_{1,3}\vee\neg v_{2,2}\vee\neg v_{3,1}\vee\neg v_{4,3}) \wedge$
$(\neg v_{1,3}\vee\neg v_{2,2}\vee\neg v_{3,2}\vee\neg v_{4,1}) \wedge (\neg v_{1,3}\vee\neg v_{2,2}\vee\neg v_{3,2}\vee\neg v_{4,2}) \wedge (\neg v_{1,3}\vee\neg v_{2,2}\vee\neg v_{3,2}\vee\neg v_{4,3}) \wedge (\neg v_{1,3}\vee\neg v_{2,2}\vee\neg v_{3,3}\vee\neg v_{4,1}) \wedge$
$(\neg v_{1,3}\vee\neg v_{2,2}\vee\neg v_{3,3}\vee\neg v_{4,2}) \wedge (\neg v_{1,3}\vee\neg v_{2,3}\vee\neg v_{3,1}\vee\neg v_{4,1}) \wedge (\neg v_{1,3}\vee\neg v_{2,3}\vee\neg v_{3,1}\vee\neg v_{4,2}) \wedge (\neg v_{1,3}\vee\neg v_{2,3}\vee\neg v_{3,1}\vee\neg v_{4,3}) \wedge$
$(\neg v_{1,3}\vee\neg v_{2,3}\vee\neg v_{3,2}\vee\neg v_{4,1}) \wedge (\neg v_{1,3}\vee\neg v_{2,3}\vee\neg v_{3,2}\vee\neg v_{4,2}) \wedge (\neg v_{1,3}\vee\neg v_{2,3}\vee\neg v_{3,2}\vee\neg v_{4,3}) \wedge (\neg v_{1,3}\vee\neg v_{2,3}\vee\neg v_{3,3}\vee\neg v_{4,1}) \wedge$
$(\neg v_{1,3}\vee\neg v_{2,3}\vee\neg v_{3,3}\vee\neg v_{4,2}) \wedge (\neg v_{1,3}\vee\neg v_{2,3}\vee\neg v_{3,3}\vee\neg v_{4,3}) \wedge$ ... 199 similar clauses... $\wedge (\neg v_{9,3}\vee\neg v_{10,3}\vee\neg v_{11,3}\vee\neg v_{12,3})$

Order Encoding

Solution of
order encoding
by MiniSAT

compared to
direct encoding



$(\neg x_{1,\le 1} \lor x_{1,\le 2}) \land (\neg x_{2,\le 1} \lor x_{2,\le 2}) \land (\neg x_{3,\le 1} \lor x_{3,\le 2}) \land (\neg x_{4,\le 1} \lor x_{4,\le 2}) \land (\neg x_{5,\le 1} \lor x_{5,\le 2}) \land (\neg x_{6,\le 1} \lor x_{6,\le 2}) \land$
$(\neg x_{7,\le 1} \lor x_{7,\le 2}) \land (\neg x_{8,\le 1} \lor x_{8,\le 2}) \land (\neg x_{9,\le 1} \lor x_{9,\le 2}) \land (\neg x_{10,\le 1} \lor x_{10,\le 2}) \land (\neg x_{11,\le 1} \lor x_{11,\le 2}) \land (\neg x_{12,\le 1} \lor x_{12,\le 2}) \land$

$(\neg x_{3,\le 1} \lor \neg x_{4,\le 1}) \land (x_{2,\le 1} \lor \neg x_{3,\le 2} \lor \neg x_{4,\le 1}) \land (x_{2,\le 1} \lor \neg x_{3,\le 1} \lor \neg x_{4,\le 2}) \land (x_{2,\le 2} \lor \neg x_{4,\le 1}) \land (x_{2,\le 2} \lor \neg x_{3,\le 2} \lor \neg x_{4,\le 2}) \land (x_{2,\le 2} \lor \neg x_{3,\le 1}) \land (x_{1,\le 1} \lor \neg x_{3,\le 2} \lor \neg x_{4,\le 1}) \land$
$(x_{1,\le 1} \lor \neg x_{3,\le 1} \lor \neg x_{4,\le 2}) \land (x_{1,\le 1} \lor x_{2,\le 1} \lor \neg x_{4,\le 1}) \land (x_{1,\le 1} \lor x_{2,\le 1} \lor \neg x_{3,\le 2} \lor \neg x_{4,\le 2}) \land (x_{1,\le 1} \lor x_{2,\le 1} \lor \neg x_{3,\le 1}) \land (x_{1,\le 1} \lor x_{2,\le 2} \lor \neg x_{4,\le 2}) \land (x_{1,\le 1} \lor x_{2,\le 2} \lor \neg x_{3,\le 2}) \land (x_{1,\le 2} \lor \neg x_{4,\le 1}) \land$
$(x_{1,\le 2} \lor \neg x_{3,\le 2} \lor \neg x_{4,\le 2}) \land (x_{1,\le 2} \lor \neg x_{3,\le 1}) \land (x_{1,\le 2} \lor x_{2,\le 1} \lor \neg x_{4,\le 2}) \land (x_{1,\le 2} \lor x_{2,\le 1} \lor \neg x_{3,\le 2}) \land (x_{1,\le 2} \lor x_{2,\le 2}) \land$

There is no clear theoretical explanation (yet) for this good performance...

Hyper-resolution and SAT-solvers

Theorem: The k-consistency closure of a CSP instance is empty *if and only if* its direct encoding has a hyper-resolution refutation of width $\leq$ k.

Theorem: If a set of non-empty clauses over n Boolean variables has a hyper-resolution refutation of width k and length m, then the ***expected number of restarts*** required by a standard randomised SAT-solver using the Decision learning scheme to discover that they are unsatisfiable is $< m \binom{n}{k}$.

$$x_1 \vee x_2 \vee \ldots \vee x_r \vee C_0$$
$$\neg x_1 \vee C_1$$
$$\neg x_2 \vee C_2$$
$$\vdots$$
$$\neg x_r \vee C_r$$
$$\overline{}$$
$$C_0 \vee C_1 \vee C_2 \vee \ldots \vee C_r$$

Theorem: If a CSP instance can be solved by establishing k-consistency, then the ***expected number of restarts*** required by a standard randomised SAT-solver on its direct encoding is $O(n^{2k}d^{2k})$.

??????????????? and SAT-solvers

Theorem: The k-consistency closure of a CSP instance is empty *if and only if* its ~~direct~~ encoding has a ~~hyper-resolution refutation~~ of width $\leq$ k.

order ????????

Theorem: If a set of non-empty clauses over n Boolean variables has a **??????????????** of width k and length m, then the ***expected number of restarts*** required by a standard randomised SAT-solver using the Decision learning scheme to discover that they are unsatisfiable is < **?????**.

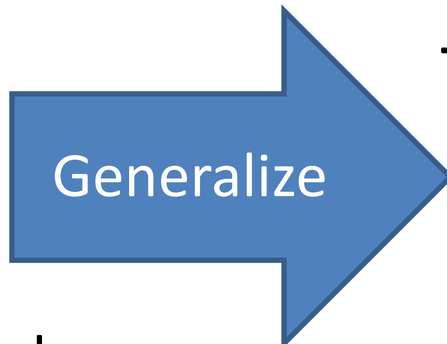
Theorem: If a CSP instance can be solved by establishing k-consistency, then the ***expected number of restarts*** required by a standard randomised SAT-solver on its **order** encoding is **???????**.

Encoding Tractable CSP Languages

"The Order Encoding: From Tractable CSP to Tractable SAT"
Justyna Petke, Peter Jeavons, SAT 2011, pp371-372.

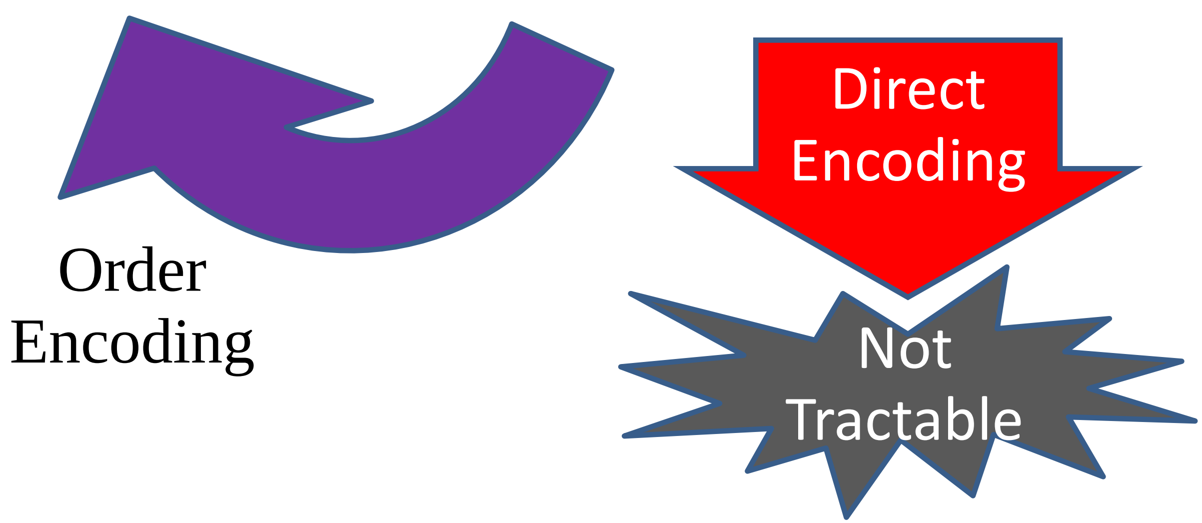
There are 5 tractable languages for SAT formed of clauses:

1. 2-clauses
2. Horn clauses
3. Anti-Horn clauses
4. Clauses with a +ve literal
5. Clauses with a -ve literal



There are 5 tractable languages for CSP that generalise these:

1. Connected row-convex
2. Max-closed
3. Min-closed
4. Constant-d closed
5. Constant-1 closed



Summary

Simple structure is not always exploited even by sophisticated solvers

Hidden structure can be exploited by simple clause-learning SAT-solvers

Learning is a powerful and flexible mechanism to exploit hidden structure

Open Questions

When does the order encoding work well? Is there an inference rule?

Can constraint solvers use learning to detect hidden structure effectively?

Is there a more flexible (hyper)graph property than bounded width?