# Exercises on Branching Time Model Checking

**Exercise 1** Simulation

Show that simulation is a transitive relation: Given any 3 Kripke structures $K_1 = (S_1, I_1, R_1, L_1)$, $K_2 = (S_2, I_2, R_2, L_2)$ and $K_3 = (S_3, I_3, R_3, L_3)$ such that $K_1 \leq K_2$ and $K_2 \leq K_3$, it holds that $K_1 \leq K_3$.

**Exercise 2** Simulation and Bisimulation

Let $M_1 = (S_1, I_1, R_1, L_1)$ and $M_2 = (S_2, I_2, R_2, L_2)$ be two Kripke structures.
A relation $H' \subseteq S_1 \times S_2$ is a *bisimulation relation* if for each $(s, s') \in H'$ holds:

- $L_1(s) = L_2(s')$,
- for each $(s,t) \in R_1$ there is a $(s',t') \in R_2$ such that $(t,t') \in H'$, and
- for each $(s',t') \in R_2$ there is a $(s,t) \in R_1$ such that $(t,t') \in H'$.

$M_1$ and $M_2$ are *bisimilar* if there is a bisimulation relation $H' \subseteq S_1 \times S_2$ *such that*

- for each initial state $s \in I_1$ there is an initial state $s' \in I_2$ with $(s,s') \in H'$, and
- for each initial state $s' \in I_2$ there is an initial state $s \in I_1$ with $(s,s') \in H'$.

In the following, we say that $H'$ *witnesses the bisimilarity of* $M_1$ *and* $M_2$ in case $H'$ is a bisimulation relation between $M_1$ and $M_2$ that satisfies the conditions stated above.

A Kripke structure $M = (S, I, R, L)$ over a set of atomic predicates $AP$ is called $AP$-*deterministic*, if

- for all $A \subseteq AP$ we have $|I \cap \{s \mid L(s) = A\}| \leq 1$, and
- for all $s \in S$ we have that $(s,t_1) \in R$, $(s,t_2) \in R$ and $L(t_1) = L(t_2)$ imply $t_1 = t_2$.

We define a sequence of relations $H_n$, for $n \geq 0$:

- $H_0 = \{(s,s') \mid s \in I_1, s' \in I_2, L_1(s) = L_2(s')\}$
- $H_{n+1} = H_n \cup \{(t,t') \mid \exists (s,s') \in H_n.(s,t) \in R_1, (s',t') \in R_2, L_1(t) = L_2(t')\}$

Finally, we define the relation $H$ as follows:

$$H = \bigcup_{n \geq 0} H_n$$

(a) Assume that $M_1$ and $M_2$ are bisimilar and AP-deterministic Kripke structures. Prove that $H$ is a bisimulation relation. Further prove that $H$ is the smallest bisimulation relation that witnesses the bisimilarity of $M_1$ and $M_2$.

*Hint:* Let $H'$ be a bisimulation relation that witnesses the bisimilarity of $M_1$ and $M_2$. Show that, for all $n \geq 0$, $H_n \subseteq H'$ holds and use this fact to show that $H$ satisfies the conditions of a bisimulation relation.

(b) Assume that $M_1$ and $M_2$ are bisimilar. Prove that, in general, $H$ is not a bisimulation relation.

(c) Let $M_1 = (S_1, I_1, R_1, L_1)$ and $M_2 = (S_2, I_2, R_2, L_2)$ be two models that are $AP$-deterministic. Show that

$$M_1 \leq M_2 \text{ and } M_1 \geq M_2 \text{ if and only if } M_1 \equiv M_2.$$