Automated Theorem Proving

and some Applications to Verification

Laura Kovács
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Homework Exercises – Automated Theorem Proving, L. Kovács

Problem 1. Establish the unsatisfiability of the following set of four formulas, using the superposition inference system $\mathbb{SRF}$:

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\begin{array}{ll}
(1) & c = d \\
(2) & f(d) \neq d \vee a = b \\
(3) & f(c) = d \\
(4) & g(a, b) \neq g(b, a)
\end{array}
$$

Problem 2. The limit of an $\mathbb{I}$-inference process $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ is the set of formulas $\bigcup_i S_i$. In other words, the limit is the set of all derived formulas.

Suppose that we have an infinite inference process such that S_0 is unsatisfiable and we use the ground superposition inference system $\mathbb{SRF}$.

Question: does completeness of $\mathbb{SRF}$ imply that the limit of the process contains the empty clause? Justify your answer!

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\begin{array}{lll}
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Question: does completeness of $\mathbb{SRF}$ imply that the limit of the process contains the empty clause? Justify your answer!

Saturation Algorithm: Fairness

Let $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be an inference process with the limit S_∞.
The process is called fair if for every $\mathbb{I}$-inference

$$\frac{F_1 \quad \ldots \quad F_n}{F} \ ,$$

if $\{F_1, \ldots, F_n\} \subseteq S_\infty$, then there exists i such that $F \in S_i$.

$\mathbb{SRF}$ completeness, reformulated

Theorem The following conditions are equivalent.

1. $\mathbb{SRF}$ is complete.
2. Let S_0 be an unsatisfiable set of formulas and we have a fair $\mathbb{SRF}$-inference process with the initial set S_0. Then the limit of this inference process contains $\square$.

Outline

The Ground Superposition System $\mathbb{SRF}$

Binary resolution inferences can be represented using derivations in the superposition system.

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  Not really. It has too many inferences. For example, from the clause $f(a) = a$ we can derive any clause of the form

$$f^m(a) = f^n(a)$$

  where $m, n \geq 0$.
  Worst of all, the derived clauses can be much larger than the original clause $f(a) = a$.

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  - Orderings;
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Recall:

- Literal: either an atom A or its negation $\neg A$.
- Clause: a disjunction $L_1 \vee \ldots \vee L_n$ of literals, where $n \geq 0$.
- Empty clause, denoted by $\square$: clause with 0 literals, that is, when $n = 0$.
- A formula in Clausal Normal Form (CNF): a conjunction of clauses.

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We denote selected literals by underlining them, e.g.,

$$\underline{f(a) = a} \vee b = c$$

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Consider this set of clauses:

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(1) & q \neq \top \ \vee \ \underline{r = \top} \\
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It is unsatisfiable:

(8)	$q = \top \ \vee \ p = \top$	$(6, 7)$
(9)	$q = \top$	$(2, 8)$
(10)	$r = \top$	$(1, 9)$
(11)	$q \neq \top$	$(3, 10)$
(12)	$\square$	$(9, 11)$

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\end{array}
$$

However, any inference with selection applied to this set of clauses gives either a clause in this set, or a clause containing a clause in this set.

Term and Literal Orderings

Term orderings

We take any ordering $\succ$ on terms such that:

1. $\succ$ is well-founded: no infinite decreasing chain $l_0 \succ l_1 \succ l_2 \succ \ldots$;
2. $\succ$ is monotonic: if $l \succ r$, then $s[l] \succ s[r]$;
3. $\succ$ is stable under substitutions: if $l \succ r$, then $l\theta \succ r\theta$.

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Term orderings: simplification ordering

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Literal orderings on equalities

Equality atom comparison treats $s = t$ as the multiset $\dot{\{}s, t\dot{\}}$.

- $(s' = t') \succ_{lit} (s = t)$ if $\dot{\{}s', t'\dot{\}} \succ \dot{\{}s, t\dot{\}}$.
- $(s' \neq t') \succ_{lit} (s \neq t)$ if $\dot{\{}s', t'\dot{\}} \succ \dot{\{}s, t\dot{\}}$.

All non-equality literals are greater than all equality literals.

Orderings and Well-Behaved Selection Functions

A literal selection function is well-behaved if

- If all selected literals are positive, then all maximal (w.r.t. $\succ$) literals in C are selected.

In other words, either a negative literal is selected, or all maximal literals must be selected.

Completeness of Superposition with Selection

Superposition with selection is complete for every well-behaved selection function.

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Consider our previous example:

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(2) $p \neq \top \vee \underline{q = \top}$

(3) $r \neq \top \vee \underline{q \neq \top}$

(4) $q \neq \top \vee \underline{p \neq \top}$

(5) $p \neq \top \vee \underline{r \neq \top}$

(6) $r \neq \top \vee \underline{p = \top}$

(7) $r = \top \vee q = \top \vee \underline{p = \top}$

A well-behave selection function must satisfy:

1. $r \succ q$, because of (1)
2. $q \succ p$, because of (2)
3. $p \succ r$, because of (6)

There is no ordering that satisfies these conditions.

Ground Superposition Inference System $\mathbb{S}\mathrm{up}_{\succ,\sigma}$

Let σ be a literal selection function.

Superposition: (right and left)

$$\frac{\underline{l = r} \lor C \quad \underline{s[l] = t} \lor D}{s[r] = t \lor C \lor D} \text{ (Sup)}, \quad \frac{\underline{l = r} \lor C \quad \underline{s[l] \neq t} \lor D}{s[r] \neq t \lor C \lor D} \text{ (Sup)},$$

where (i) $l \succ r$, (ii) $s[l] \succ t$, (iii) $l = r$ is strictly greater than any literal in C, (iv) $s[l] = t$ is greater than or equal to any literal in D.

Equality Resolution:

$$\frac{\underline{s \neq s} \lor C}{C} \text{ (ER)},$$

Equality Factoring:

$$\frac{\underline{s = t} \lor s = t' \lor C}{s = t \lor t \neq t' \lor C} \text{ (EF)},$$

where (i) $s \succ t \succeq t'$; (ii) $s = t$ is greater than or equal to any literal in C.

Non-Ground Superposition Rule

Superposition:

$$\frac{\underline{l = r} \lor C \quad \underline{s[l'] = t} \lor D}{(s[r] = t \lor C \lor D)\theta} \text{ (Sup)}, \qquad \frac{\underline{l = r} \lor C \quad \underline{s[l'] \neq t} \lor D}{(s[r] \neq t \lor C \lor D)\theta} \text{ (Sup)},$$

where

1. θ is an mgu of l and l';
2. l' is not a variable;
3. $r\theta \not\succeq l\theta$;
4. $t\theta \not\succeq s[l']\theta$.
5. ...

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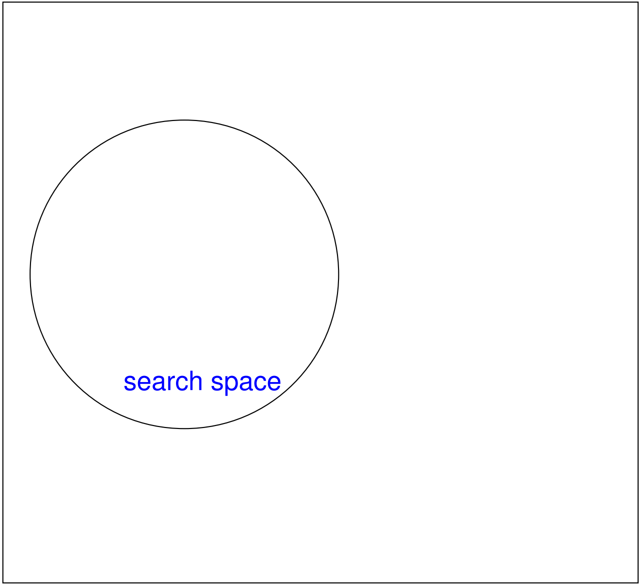
Observation:

- ordering is partial, hence conditions like $r\theta \not\succeq l\theta$;

Fair Saturation Algorithms: Inference Selection by Clause Selection



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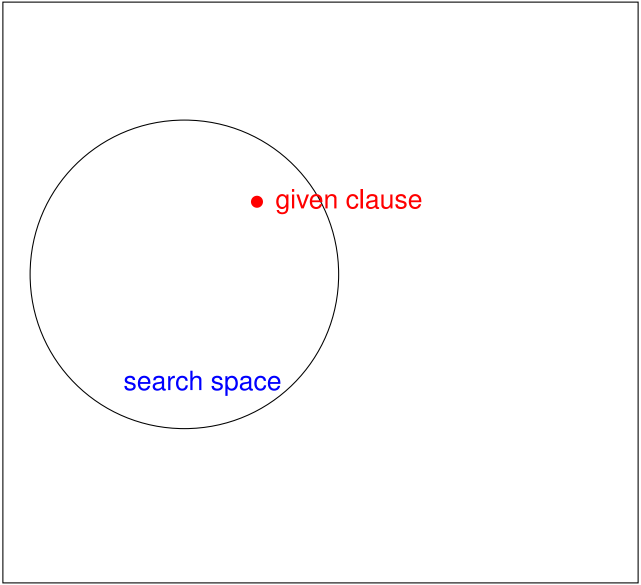

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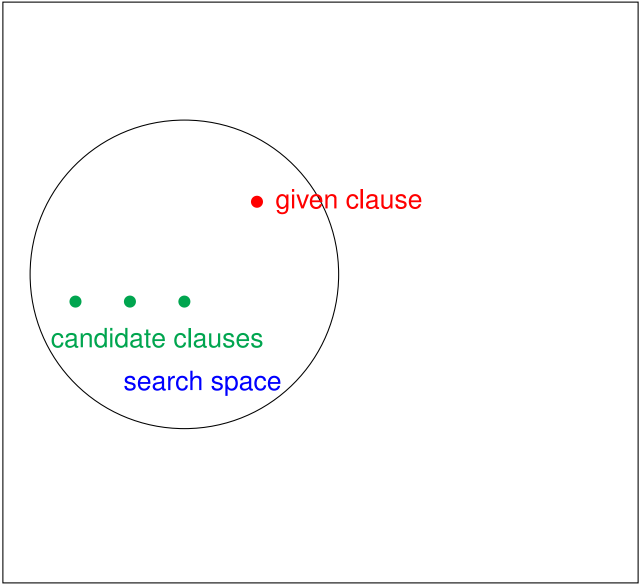

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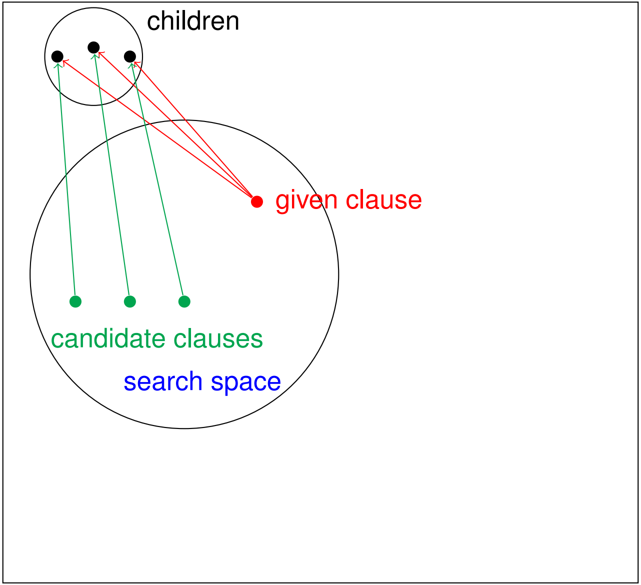

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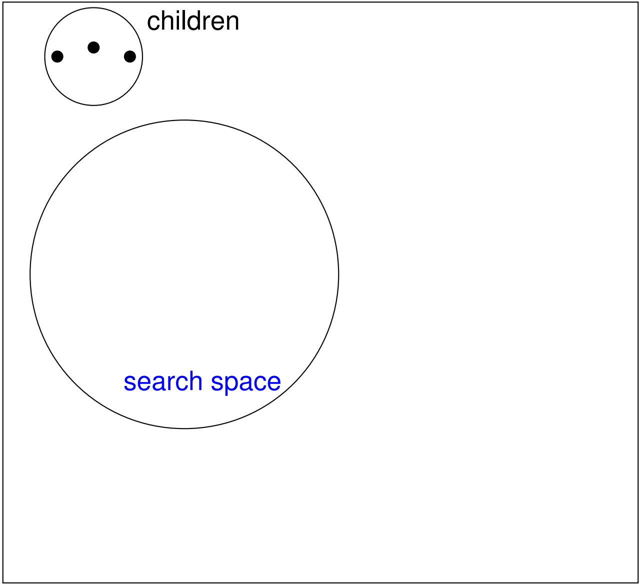

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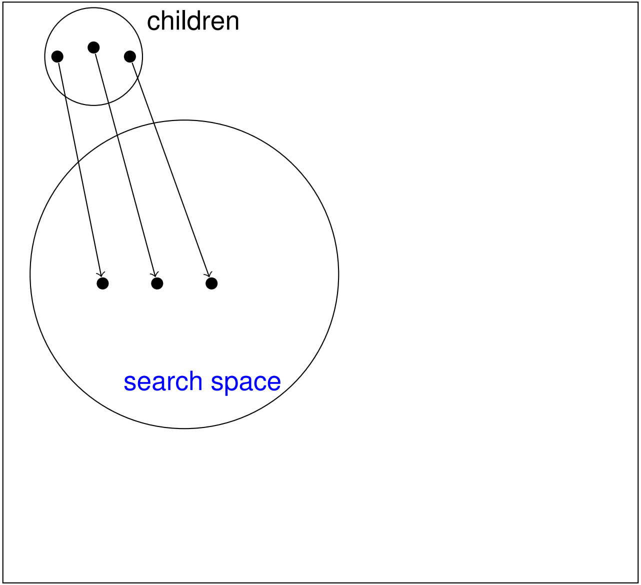

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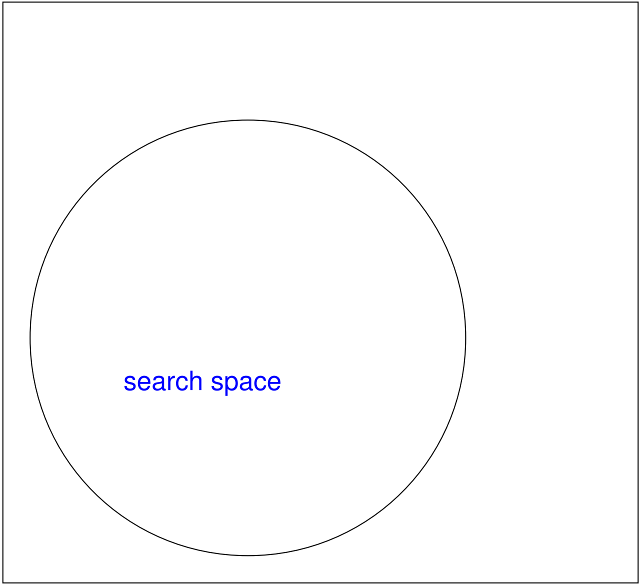

Saturation Algorithm

Even when we implement inference selection by clause selection, there are too many inferences, especially when the search space grows.

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Solution: only apply inferences to the selected clause and the previously selected clauses.

Thus, the search space is divided in two parts:

- active clauses, that participate in inferences;
- passive clauses, that do not participate in inferences.

Redundancy

A clause $C \in S$ is called redundant in S if it is a logical consequence of clauses in S strictly smaller than C.

Examples

A tautology $p \vee \neg p \vee C$ is a logical consequence of the empty set of formulas:

$$\models p \vee \neg p \vee C,$$

therefore it is redundant.

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We know that C subsumes $C \vee D$. Note

$$C \vee D \succ C$$
$$C \models C \vee D$$

therefore subsumed clauses are redundant.

Redundant Clauses Can be Removed

In $\mathbb{SRF}$ redundant clauses can be removed from the search space.

Checking Redundancy

Suppose that the current search space S contains no redundant clauses. How can a redundant clause appear in the inference process?

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Only when a new clause (a child of the selected clause and possibly other clauses) is added.

Classification of redundancy checks:

- The child is redundant;
- The child makes one of the clauses in the search space redundant.

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Classification of redundancy checks:

- The child is redundant; $\rightarrow$ forward simplification
- The child makes one of the clauses in the search space redundant. $\rightarrow$ backward simplification

Summary of a Proof by Vampire: Example from Algebra (recap)

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Refutation found. Thanks to Tanya!
203. $false [subsumption resolution 202,14]
202. sP1(mult(sK,sK0)) [backward demodulation 188,15]
188. mult(X8,X9) = mult(X9,X8) [superposition 22,87]
87. mult(X2,mult(X1,X2)) = X1 [forward demodulation 71,27]
71. mult(inverse(X1),e) = mult(X2,mult(X1,X2)) [superposition 23,20]
27. mult(inverse(X2),e) = X2 [superposition 22,10]
23. mult(inverse(X4),mult(X4,X5)) = X5 [forward demodulation 18,9]
22. mult(X0,mult(X0,X1)) = X1 [forward demodulation 16,9]
20. e = mult(X0,mult(X1,mult(X0,X1))) [superposition 11,12]
18. mult(e,X5) = mult(inverse(X4),mult(X4,X5)) [superposition 11,10]
16. mult(e,X1) = mult(X0,mult(X0,X1)) [superposition 11,12]
15. sP1(mult(sK0,sK)) [inequality splitting 13,14]
14. ~sP1(mult(sK,sK0)) [inequality splitting name introduction]
13. mult(sK,sK0) != mult(sK0,sK) [cnf transformation 8]
12. e = mult(X0,X0) (0:5) [cnf transformation 4]
11. mult(mult(X0,X1),X2)=mult(X0,mult(X1,X2))[cnf transformation 3]
10. e = mult(inverse(X0),X0) [cnf transformation 2]
9. mult(e,X0) = X0 [cnf transformation 1]
8. mult(sK,sK0) != mult(sK0,sK) [skolemisation 7]
7. ? [X0,X1] : mult(X0,X1) != mult(X1,X0) [ennf transformation 6]
6. ~! [X0,X1] : mult(X0,X1) = mult(X1,X0) [negated conjecture 5]
5. ! [X0,X1] : mult(X0,X1) = mult(X1,X0) [input]
4. ! [X0] : e = mult(X0,X0)[input]
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5. ! [X0,X1] : mult(X0,X1) = mult(X1,X0) [input]
4. ! [X0] : e = mult(X0,X0)[input]
3. ! [X0,X1,X2] : mult(mult(X0,X1),X2) = mult(X0,mult(X1,X2))[input]
2. ! [X0] : e = mult(inverse(X0),X0) [input]
1. ! [X0] : mult(e,X0) = X0 [input]
```

- Proof by refutation;

Summary of a Proof by Vampire: Example from Algebra (recap)

```
Refutation found. Thanks to Tanya!
203. $false [subsumption resolution 202,14]
202. sP1(mult(sK,sK0)) [backward demodulation 188,15]
188. mult(X8,X9) = mult(X9,X8) [superposition 22,87]
87. mult(X2,mult(X1,X2)) = X1 [forward demodulation 71,27]
71. mult(inverse(X1),e) = mult(X2,mult(X1,X2)) [superposition 23,20]
27. mult(inverse(X2),e) = X2 [superposition 22,10]
23. mult(inverse(X4),mult(X4,X5)) = X5 [forward demodulation 18,9]
22. mult(X0,mult(X0,X1)) = X1 [forward demodulation 16,9]
20. e = mult(X0,mult(X1,mult(X0,X1))) [superposition 11,12]
18. mult(e,X5) = mult(inverse(X4),mult(X4,X5)) [superposition 11,10]
16. mult(e,X1) = mult(X0,mult(X0,X1)) [superposition 11,12]
15. sP1(mult(sK0,sK)) [inequality splitting 13,14]
14. ~sP1(mult(sK,sK0)) [inequality splitting name introduction]
13. mult(sK,sK0) != mult(sK0,sK) [cnf transformation 8]
12. e = mult(X0,X0) (0:5) [cnf transformation 4]
11. mult(mult(X0,X1),X2)=mult(X0,mult(X1,X2))[cnf transformation 3]
10. e = mult(inverse(X0),X0) [cnf transformation 2]
9. mult(e,X0) = X0 [cnf transformation 1]
8. mult(sK,sK0) != mult(sK0,sK) [skolemisation 7]
7. ? [X0,X1] : mult(X0,X1) != mult(X1,X0) [ennf transformation 6]
6. ~! [X0,X1] : mult(X0,X1) = mult(X1,X0) [negated conjecture 5]
5. ! [X0,X1] : mult(X0,X1) = mult(X1,X0) [input]
4. ! [X0] : e = mult(X0,X0)[input]
3. ! [X0,X1,X2] : mult(mult(X0,X1),X2) = mult(X0,mult(X1,X2))[input]
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```

- Proof by refutation;
- Each inference derives a new formula;
- Generating and simplifying inferences.

$\Big\}$ Fair saturation algorithm

Outline

The Superposition Inference System

Colored Proofs, Interpolation and Symbol Elimination

Sorts and Theories

Interpolation

Theorem

Let A, B be closed formulas and let $A \vdash B$.

Then there exists a formula I such that

1. $A \vdash I$ *and* $I \vdash B$*;*
2. *every symbol of* I *occurs both in* A *and* B*;*

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Interpolation has many uses in verification.

When we deal with refutations rather than proofs and have an unsatisfiable set $\{A, B\}$, it is convenient to use reverse interpolants of A and B, that is, a formula I such that

1. $A \vdash I$ and $\{I, B\}$ is unsatisfiable;
2. every symbol of I occurs both in A and B.

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- We have two formulas: A and B.
- Each symbol in A is either blue or green.
- Each symbol in B is either red or green.
- We know that $\vdash A \rightarrow B$.
- Our goal is to find a green formula I such that
  1. $\vdash A \rightarrow I$;
  2. $\vdash I \rightarrow B$.

Local Derivations

A derivation is called local (well-colored) if each inference in it

$$\frac{C_1 \quad \cdots \quad C_n}{C}$$

either has no blue symbols or has no red symbols.
That is, one cannot mix blue and red in the same inference.

Local Derivations: Example

- A := $\forall x(x = a)$
- B := $c = b$
- Interpolant: $\forall x \forall y(x = y)$

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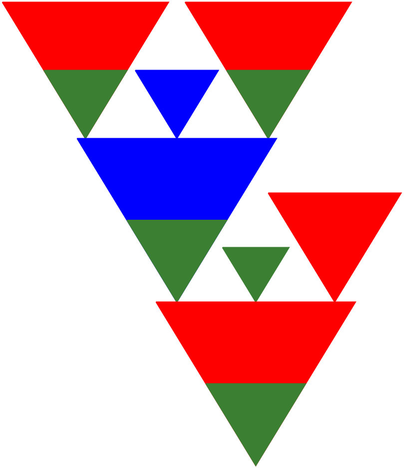
Local Proof

$$\dfrac{\dfrac{\dfrac{x=a \qquad y=a}{x=y} \qquad c\neq b}{y\neq b}}{\bot}$$

Shape of a local derivation



Symbol Eliminating Inference

- At least one of the premises is not green.
- The conclusion is green.

$$\cfrac{\boxed{\cfrac{x = a \quad y = a}{x = y}} \qquad c \neq b}{\boxed{\cfrac{y \neq b}{\perp}}}$$

Extracting Interpolants from Local Proofs

Theorem (CADE'09)

Let Π be a local refutation. Then one can extract from Π in linear time a reverse interpolant I of A and B. This interpolant is ground if all formulas in Π are ground.

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What is remarkable in this theorem:

- No restriction on the calculus (only soundness required) – can be used with theories.
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Further result: generate minimal interpolants wrt various measures. (POPL'12)

Symbol Elimination

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This technique was used in our experiments on automatic loop invariant generation (FASE'09).

Outline

How Vampire Proves Problems in Arithmetic

- adding theory axioms;
- evaluating expressions, when possible;
- (future) SMT solving.