ENS Paris-Saclay

Bachelor's Degree Research Internship

A crossing between topology and computability,

Topological analysis of represented spaces and computable maps: cb_0 spaces and non-countably-based spaces

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Contents

1 Introduction

1.1 General context

Since Alonzo Church, Alan Turing, Kurt Gödel and other scientists introduced the first notions of computability and computational models at the beginning of the 30s, numerous attempts have tried to extend the previous theoretical models of calculus on other sets, like $\mathbb{R}$, or spaces, like the maps of $\mathbb{N}^{\mathbb{N}}$.

The first examples date back to Alan Turing himself in these 30s, but a more modern reference like [Wei00] presents a commonly accepted extension of computational notions on the real line and other effective versions of metric spaces: namely, Type II computability. Several links with topology begin to emerge here: a classic illustration of this idea is that "computable" maps on those spaces are continuous for the topology induced by the representation. Furthermore, one should note that the choice of the representation has a crucial influence on the computational power of these new models.

Descriptive Set Theory turns out to be a useful notion in order to investigate the topological aspects of computability, in particular the several hierarchies (Borel hierarchy, difference hierarchy) in their effective versions. While these notions of computability and these hierarchies have been thoroughly studied in some classes of countably-based spaces (Polish spaces first chronologically, then ω-continuous domains and more recently quasi-Polish spaces in [dB13]), their extensions to more general spaces are still at a beginning stage and very little is known.

1.2 Focus of this study

The aim of this thesis is to study the relationships between topology and computability on several classes of represented spaces. The case of countably-based (or cb_0) spaces has been studied for a handful of years, but we still lack a general case for well-known theorems in subcases of the theory (like in Polish spaces). Additionally, scientists know very little of some other spaces (like the space of real polynomials $\mathbb{R}[X]$), even though the possibility to algorithmically manipulate them would be truly desirable.

My attempt here is to generalize results (which were already known in quasi-Polish spaces) to countably-based spaces, and to study whether we could envision extending notions of computability (and of Descriptive Set Theory) outside of their usual background, in the specific example of real polynomials.

1.3 My contribution

The purpose of this internship was dual. The first part was about learning the required skills to contribute to this field of knowledge. In order to do so, I avoided shorcuts and took enough time to demonstrate and rewrite by myself the proofs of all the theorems I enunciate in this paper. The second part was about mathematical approaches, with two intents: on the one hand, I reduced topological phenomenons to their essential arguments and removed the unnecessary hypotheses, on the other hand I generalized these results onto spaces where a notion of computability could be desirable.

In this document, I disclose three of the main results we have obtained:

- In countably-based spaces, the topological complexity of an *effective* set (ie. its position in the difference hierarchy) and its algorithmic complexity (ie. the complexity of its preimage under the representation) are equivalent. This is an effectivization of a result already known since [dB13].
- In countably-based spaces, it is equivalent for a map to be piecewise-computable with a countable cover, and for its "algorithmic realization" to also be piecewise-computable with a countable cover. We also generalize an already known counter-example in the finite case.
- On the space of real polynomials, we exhibit a set whose algorithmic complexity differs from its topological complexity. This implies that a representation cannot capture every topological phenomenon on $\mathbb{R}[X]$. As a consequence, we demonstrate that the Wadge and the Hausdorff-Kuratowski theorems do not apply on this space.

1.4 Arguments in favor of these results

We believe these results to be satisfactory answers, as they extend already well-known results to more general spaces and contribute to the understanding of an already substantial theory. Among these results, the study of piecewise-continuity is especially conclusive, as it solves both the countable and finite cases.

The investigations on the space of real polynomials raise the most unanswered questions. Indeed, we have yet to understand the reasons why the theory of represented spaces fails to capture the topological phenomenons on this space. Our work initiates a better understanding of this theory and calls for a characterization of the spaces it analyses correctly.

1.5 Overview and future work

During this internship, we have studied several possibilities to generalize the theory of represented spaces and its computable aspects outside of their usual backgrounds. While well-known results seem to correctly extend themselves on countably-based spaces, very little is known on non-countably-based spaces, despite our first study on the specific example of real polynomials. Indeed, we have raised a few possible explanations for the phenomenons we observed in this specific case. But we still lack a mathematical description of the differences between topological and algorithmic complexity, or even a characterization of the objects that could be studied by representations: such progress would be an immeasurable contribution to this field of study and shed a new light on the results we have just obtained.

1.6 Acknowledgements

First of all, I want to thank Mathieu Hoyrup for his friendly presence from the beginning to the end of this internship, for his precise and flawless proofreading, and for his communicative enthusiasm. Although I was not familiar with this field of study at first, he proved to be a great teacher and working with him was thoroughly enjoyable.

I am also thankful to the Mocqua team for their welcome, the discussions we had about their lives as researchers, as well as the philosophical and historical exchanges about science that frequently happened during our meals. I have learnt a lot in their company.

Finally, I want to express my gratitude to my relatives for their support in the writing of this paper, particularly Guillaume for his unplanned yet very timely proofreading and Sully for their help during the translation of this report. I hope that I have, thanks to them, successfully produced a thesis that is both accessible, intriguing and interesting.

Last but not least: I wish you a great and enjoyable reading, and I hope these pages will inspire you as much as they have inspired me.

2 Prerequisites and notations

Throughout this paper, I will suppose that the reader has some familiarity with usual notions in general topology, for example: definitions of open/closed sets, neighborhoods, closure and interior, density and more generally notions associated to the Baire category theorem, definitions of continuity and bases of topological spaces, along with separation axioms. I will also ponctually refer to some elementary definitions on ordinals and on traditional computability (Turing Machines in particular).

Aditionnaly, I will use the following notations:

- $\mathbb{N}$ the set of natural numbers.
- $\mathbb{N}^*$ the set of finite words over the alphabet $\mathbb{N}$.
- $\mathcal{N} = \mathbb{N}^{\mathbb{N}} = \mathbb{N} \mapsto \mathbb{N}$ the Baire space, indifferently seen as infinite sequences over the alphabet $\mathbb{N}$ or as maps of $\mathbb{N} \mapsto \mathbb{N}$.

- $|u|$ the length of the word $u \in \mathbb{N}^*$ (in other words, $|u| = n$ if $u \in \mathbb{N}^n$).
- $u \cdot v$ the concatenation of two words $u \in \mathbb{N}^*$ and $v \in \mathbb{N}^* \cup \mathcal{N}$.
- $u \sqsubseteq v$ (for $u \in \mathbb{N}^*$ and $v \in \mathbb{N}^* \cup \mathcal{N}$) when u is a prefix of v.
- $p|k \in \mathbb{N}^*$ the prefix of size $k+1$ ($k \in \mathbb{N}$) of the word $p \in \mathcal{N}$.
- $[\sigma] = \{p \in \mathcal{N} : \sigma \sqsubseteq p\} = \sigma \cdot \mathcal{N}$, the cylinder generated by σ (for a word $\sigma \in \mathbb{N}^*$).

- $f :\subseteq X \mapsto Y$ partial maps of domain $\mathrm{dom}(f) \subseteq X$ and of codomain Y.
- A^{c} the complement of the set A.
- π_1 and π_2 the two projections (of type $A^2 \mapsto A$, for A a set), namely $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

- O the set of computable ordinals. (cf. section D)
- $|a|_O$ (for $a \in O$) the ordinal of which a is a name.
- $<_o$ the usual strict order on the set O.

Furthermore, I will use to the following conventions:

- $(X, \mathcal{O}(X))$ or $(Y, \mathcal{O}(Y))$ for topological spaces, where $\mathcal{O}(X)$ represents the set of open sets in X.
- A topological set is cb_0 if it is both T_0 and countably-based.
- $(X, \mathcal{B})$ (or $(X, \mathcal{B}^{(X)})$ if the context is ambiguous) for a topological space equiped with a basis of its open sets.
- δ for a representation (or ponctually $\delta^{(X)}$ if the context is ambiguous), and δ_X for the standard representation of a cb_0 space $(X, \mathcal{B})$.
- Γ for complexity class in the difference or the Borel hierarchy, while $\tilde{\Gamma}$ (its dual class) is the set of A such that A^{c} belongs in Γ.

3 Computability on represented spaces

Computability traditionally focuses on functions $f :\subseteq \Sigma^* \mapsto \Sigma^*$, where Σ is a finite alphabet. There exists numerous variations of this model (for example functions $f' :\subseteq (\Sigma^*)^n \mapsto \Sigma^*$), that eventually turned out to be equivalent in terms of computational power. They are underlain with the same fundamental ideas: in order to compute functions on the integers or on rational numbers, computability traditionally uses words of Σ^* as codes/names. In this context, a (Turing) machine performs transformations on words (ie. it maps words to words), without "understanding" the mathematical meaning of these transformations (ie. which operation is actually performed on the corresponding integers).

However, functions of $f :\subseteq \Sigma^* \mapsto \Sigma^*$ are too limited to properly extend computability on uncountable sets (such as real numbers), because of cardinality: Σ^* is countable, and as such cannot be used as a set of names/codes for an uncountable set. The key idea behind Type-2 computability is to extend those notions to infinite words. As this paper does not focus on technical details, we will directly use the more abstract but equivalent model of functions $f :\subseteq \mathbb{N}^{\mathbb{N}} \mapsto \mathbb{N}^{\mathbb{N}}$ that map infinite sequences of integers with one another.

In this section, we first define computability on maps that have the Baire space as domain and codomain. By properly defining the intuitive notion of representations (which map codes/names in the Baire space onto elements of a set X), we will then obtain a notion of computability on any set having at most the cardinality of the continuum. During these developments, I will explain some links between computability notions and their topological duals which I mentioned in the introduction.

3.1 The Baire space: definition and first properties

Definition 3.1.1: the Baire space

The set $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ equipped with the product topology is called the **Baire space**. Its elements are indifferently seen as maps of type $\mathbb{N} \mapsto \mathbb{N}$ or as infinite words over the alphabet $\mathbb{N}$.

Regarding its topology, one should first notice that for any $\sigma \in \mathbb{N}^*$ and $p \in \mathcal{N}$ such that $\sigma \sqsubseteq p$, the cylinder $[\sigma] = \{f \in \mathcal{N} : \sigma \sqsubseteq f\}$ is an open (by the definition of the product topology) neighborhood of p. One easily deduces that the set of cylinders $\{[\sigma] : \sigma \in \mathbb{N}^*\}$ is a (countable) basis of $\mathcal{N}$.

Secondly, $\mathcal{N} \setminus [\sigma] = \bigcup_{i \in \mathbb{N}} \{[\tau] : \tau \in \mathbb{N}^*, |\tau| \geq i, \text{ and } \tau(i) \neq \sigma(i)\}$ is also an open set, which proves that cylinders are both open and closed (ie. clopen) sets. As a consequence, $\mathcal{N}$ is a zero-dimensional space, and is furthermore totally discontinuous.

Finally, as $\mathbb{N}$ is a countable set equipped with the discrete topology, $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ is a Polish space.

3.2 Computability and Turing machines

3.2.1 Definition

To define a notion of computability on $\mathcal{N}$, we could use Turing machines mapping infinite words together like we mentioned in the introduction: this model (Type-2 computability) is perfectly functional, and has already been extensively studied, for example in [Wei00]. Here, as I do not intend to examine technical aspects (like encodings) in great detail, I choose to use another model, equivalent to the previous one, closer to [Wei85]:

Definition 3.2.1: Computability of maps $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$

A map $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$ is **computable** if there exists a Turing machine M such that:

$$\forall p \in \text{dom}(F), \forall n \in \mathbb{N}, \quad M^p(n) = F(p)(n)$$

where M^p is the Turing machine M equipped with the oracle p.

Informally, this notion is equivalent to the one of [Wei00]. It seems indeed equivalent have an infinite word as an input or to use it as an oracle tape. Morover, the computability of the function $F(p)$ in each $n \in \mathbb{N}$ enables us to enumerate the sequence $(F(p)(n))_{n \in \mathbb{N}}$ without ever stopping the computations or the enumeration. This is as good as “computing” the infinite word $F(p)$ and writing it on an infinite output tape letter by letter, which corresponds to the intuitive notion of “computation on infinite words”.

3.2.2 Computability and continuity

Before proceeding with the next definitions, it would be relevant to improve our grasp of the previous notions by trying to solve this question: is there a characterization of computable maps, among the maps of type $\mathcal{N} \mapsto \mathcal{N}$? This is the first time we notice links between topology and computability. Indeed, inspired by [Wei00]'s methods, we prove that computable maps are continuous, and that continuous maps are computable, up to an additional oracle.

Definition 3.2.2: Effective continuity of maps $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$

A map $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$ is **effectively continuous** if there exists a recursively enumerable set $\mathcal{A} \subseteq \{(\sigma, \tau) \in (\mathbb{N}^*)^2 : [\tau] \subseteq F^{-1}([\sigma])\}$ such that:

$$\text{for all } \sigma \in \mathbb{N}^*,\ F^{-1}([\sigma]) = \bigcup_{e \in \mathcal{A}:\pi_1(e)=\sigma} \pi_2(e).$$

Property 3.2.3: Computability $\iff$ Effective continuity

Let $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$ be a map. Then F is computable if and only if it is effectively continuous.

Property 3.2.4: Computability with oracle $\iff$ Continuity

Let $F : \mathcal{N} \mapsto \mathcal{N}$. Then F is continuous if and only if there exists an oracle $\mathcal{A} \subseteq \mathbb{N}$ such that F is computable relatively to $\mathcal{A}$. In other words, if and only if there exists a Turing machine M and a set $\mathcal{A} \subseteq \mathbb{N}$ such that: $\forall p \in \text{dom}(F), \forall i \in \mathbb{N} \quad F(p)(i) = M^{\mathcal{A},p}(i)$

3.3 Representations and represented spaces

Now that we have a notion of computability on the Baire space, we define the notion of "names" (or "codes") of a topological space (like in [KW85]), which will later enable us to develop a notion of computability on many spaces.

3.3.1 Definitions

Definition 3.3.1: Representations and represented spaces

Let X be a set (with cardinality at most the continuum).

1. A **representation** of X is a surjective map $\delta :\subseteq \mathcal{N} \mapsto X$. For $x \in X$, the "names" of x are the elements $p \in \delta^{-1}(x)$.
2. The pair (X, δ) is called a **represented space**.

There are two possibilities to define a *topological* represented space.

1. The first one consists in equipping the represented set (X, δ) with the final topology with respect to δ. In other words, to equip X with the topology

$$\mathcal{O}(X) = \left\{O \subseteq X : \delta^{-1}(O) \in \mathcal{O}(\text{dom}(\delta))\right\}$$

2. The second one consists in equipping a topological space with a representation that "respects" the topology. To do so, we introduce the notion of admissible representation, with a formalism that slightly differs from [KW85] (but is equivalent):

Definition 3.3.2: Admissible representation

Let (X, δ) be a represented space. The representation δ is **admissible** if it is both continuous and such that :

for any continuous map $f :\subseteq \mathcal{N} \mapsto X$, there exists another continuous map $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$ verifying: $\boldsymbol{\forall p \in \text{dom}(f) \quad f(p) = \delta \circ F(p)}$

3.3.2 Computability and adjacent notions in represented spaces

Now that we have a correct notion of a naming system for any space, we can define a notion of computability on these spaces. We proceed below according to [Wei00]'s approach:

Definition 3.3.3: $(\delta^{(X)}, \delta^{(Y)})$-continuity and $(\delta^{(X)}, \delta^{(Y)})$-computability

Let $(X, \delta^{(X)})$ and $(Y, \delta^{(Y)})$ be two represented spaces. A map $f :\subseteq X \mapsto Y$ is

$(\boldsymbol{\delta^{(X)}}, \boldsymbol{\delta^{(Y)}})$-continuous (resp. **$(\boldsymbol{\delta^{(X)}}, \boldsymbol{\delta^{(Y)}})$-computable**)

if there exists a continuous (resp. computable) map $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$ (a **realization of $\boldsymbol{f}$**), such that:

$$\forall p \in \text{dom}(f \circ \delta^{(X)}) \qquad f \circ \delta^{(X)}(p) = \delta^{(Y)} \circ F(p)$$

$$\begin{array}{ccc} X & \xrightarrow{\;\boldsymbol{f}\;} & Y \\ {\scriptstyle \delta^{(X)}}\uparrow & & \uparrow{\scriptstyle \delta^{(Y)}} \\ \mathcal{N} & \xrightarrow{\;\exists \boldsymbol{F}\;} & \mathcal{N} \end{array}$$

The computational power of a computable map depends, in general, of the representation one chooses. Below, we will see with property 4.2.3 that it is possible to forget this dependence in the context of admissible representations on cb_0 spaces.

4 Effective cb_0 topological spaces

Descriptive Set Theory is the study of some classes of sets, traditionally in the context of Polish spaces. Some ulterior motivations (mainly from theoretical computer science) extended it to ω-continuous domains (cf. [Sel06]), and more recently to the so-called "quasi-Polish" spaces (cf. [dB13]). Here, we detail an extension to cb_0 spaces based on works like [HRSS19], [Sch02] or [Sel08].

4.1 First definitions

Unless stated otherwise, all the following definitions are inspired from [HRSS19].

Definition 4.1.1: Effective cb_0 topological space

An **effective $\textbf{cb}_0$ topological space** is a T_0 topological space $(X, \mathcal{O}(X))$ that has a countable basis $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ on which any finite intersection is computable (such a notion is explained below).

A "computable" intersection is defined as follows. Let $W = \{W_e\}_{e \in \mathbb{N}}$ be a fixed enumeration of the recursively enumerable sets of $\mathbb{N}$. By a "computable" finite intersection of open sets, we mean there exists a computable function $f : \mathbb{N}^2 \mapsto \mathbb{N}$ such that:

$$\forall i, j \in \mathbb{N}, B_i \cap B_j = \bigcup_{k \in W_{f(i,j)}} B_k$$

We now provide an effectivization for the notion of open sets:

Definition 4.1.2: Effective open sets

Let $(X, \mathcal{B})$ be an effective cb_0 space. An open set $U \in \mathcal{O}(X)$ is **effective** if there exists a recursively enumerable set $W \subseteq \mathbb{N}$ such that $U = \bigcup_{k \in W} B_k$.
From now on, $\mathcal{O}_{\text{eff}}(X)$ will be the effective open sets of X.

Finally, we define "effectively continuous" maps between cb_0 spaces (they are the "computable maps" of [HRSS19]) in the following way, by analogy with topological continuity:

Definition 4.1.3: Effective continuity

Let $(X, \mathcal{B}^{(X)})$ and $(Y, \mathcal{B}^{(Y)})$ be two effective cb_0 spaces. A map $f : X \mapsto Y$ is **effectively continuous** if the family of sets $\left\{f^{-1}(B_i^{(Y)})\right\}_{i\in\mathbb{N}}$ is a *uniform* family of effective open sets.

The careful reader would notice that we have used a notion of "uniformity" in the previous definition, which is yet to be explained. This notion is not related to topology by any mean, and is only a computational restriction that we can define in the following way:

Definition 4.1.4: Uniformity

An **"effective uniform family"** is a family of objects such that there exists a *single* Turing machine that recursively enumerates them.

In the context of the previous definition, this means there exists a single Turing machine that recursively enumerates some pairs (i, k) such that $f^{-1}(B_i^{(Y)}) = \bigcup_{k\in W} B_k$ (for each $i \in \mathbb{N}$).

Given an effective cb_0 space, we now would like to create an admissible representation that respects its topology. By combining the T_0 axiom and a countable basis, we define the standard representation, that identifies a point by enumerating the open sets this point belongs in:

Definition 4.1.5: Standard representation

Let $(X, \mathcal{B})$ be an effective cb_0 space. The **standard representation $\boldsymbol{\delta_X}$** is the map defined as $\delta_X = e^{-1} \circ \rho_{|A}$, where:

$$\begin{aligned} \mathrm{e}:\ & x \in X \mapsto \{i \in \mathbb{N} : x \in B_i\} \in \mathcal{P}(\mathbb{N}) \\ & \quad \text{is an homeomorphism between } X \text{ and its image} \\ \rho:\ & p \in \mathcal{N} \mapsto \{i \in \mathbb{N} : \exists n, p(n) = i+1\} \in \mathcal{P}(\mathbb{N}) \\ A =\ & \rho^{-1}(e(X)) \end{aligned}$$

Such a representation has the following properties:

Property 4.1.6: Properties of the standard representation

The standard representation δ_X of an effective cb_0 space $(X, \mathcal{B})$ is:

1. admissible
2. effectively continuous
3. effectively open
4. and it computably realizes effectively continuous maps.

In the rest of the paper, any result in cb_0 spaces will use the standard representation. However, we assert they remain valid if the representation is effectively admissible, ie. equivalent by an effective continuous map to the standard representation.

4.2 Properties of the standard representation

As we did before, we should first take some time to familiarize with the topological and computational properties of the standard representation. Let us begin by recalling a characterization of its topology one can read in [BH02]:

Property 4.2.1: Admissible representations and final topology

Let $(X, \mathcal{B})$ be a cb_0 topological space. For any admissible representation δ of X, the final topology with respect to δ is $\mathcal{O}(X)$.

Furthermore, we claimed that the notions of $(\delta^{(X)}, \delta^{(Y)})$-continuity and computability were intrinsically related to the choice of the representations. In the case of the standard representations, we show that these notions coincide with their topological (cf. [BH02]) and computable (cf. [HRSS19]) equivalents:

Property 4.2.2: (δ_X, δ_Y)-continuity and topological continuity

Let $(X, \mathcal{B}^{(X)})$ and $(Y, \mathcal{B}^{(Y)})$ be two cb_0 topological spaces. For any map $f :\subseteq X \mapsto Y$, the two following conditions are equivalent:

1. f is continuous.
2. f is (δ_X, δ_Y)-continuous.

Property 4.2.3: (δ_X, δ_Y)-computability and effective continuity

Let $(X, \mathcal{B}^{(X)})$ and $(Y, \mathcal{B}^{(Y)})$ be two effective cb_0 topological spaces. For any map $f :\subseteq X \mapsto Y$, the two following conditions are equivalent:

1. f is (δ_X, δ_Y)-computable.
2. f is effectively continuous.

4.3 Descriptive Set Theory on cb_0 spaces

We now introduce the Borel and the difference hierarchies. Then, we will be thouroughly prepared to explore the main results of this internship.

4.3.1 The Borel hierarchy

The Borel hierarchy is a stratification of the Borel σ-algebra generated by the open sets, which assigns a countable ordinal as a rank to each Borel set. This will serve as a notion of complexity for Borel sets, and is formalized in the following definition (for example used in [Sel08]):

Definition 4.3.1: Boldface Borel hierarchy

Let $(X, \mathcal{O}(X))$ be a topological space. The Boldface Borel hierarchy is defined as follows:

$$\boldsymbol{\Sigma}^0_1(X) = \mathcal{O}(X) \qquad \text{and for any ordinal } 2 \leq \beta < \omega_1 :$$

$$\boldsymbol{\Sigma}^0_\beta(X) = \left\{ \bigcup_{n \in \mathbb{N}} A_n \setminus A'_n : A_n, A'_n \in \boldsymbol{\Sigma}^0_{\beta_n}(X) \text{ and } \beta_n < \beta \right\}$$

We also define the dual and intersection classes for $1 \leq \beta < \omega_1$:

$$\boldsymbol{\Pi}^0_\beta(X) = \left\{ A^c : A \in \boldsymbol{\Sigma}^0_\beta(X) \right\} \qquad \boldsymbol{\Delta}^0_\beta(X) = \left\{ A : A \in \boldsymbol{\Sigma}^0_\beta(X) \cap \boldsymbol{\Pi}^0_\beta(X) \right\}$$

Let me underline that this definition of $\boldsymbol{\Sigma}^0_\beta$ sets is slightly different from the usual definition on Polish spaces, that one can read in [Kec95]. Indeed, any open set can be written as the countable union of closed sets on metric spaces. As such, the Borel hierarchy defined by $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}(X) = \left\{\bigcup_{n\in\mathbb{N}} A_n : A_n \in \boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\beta_n}}(X), \beta_n < \beta\right\}$, can truly be called a hierarchy (as $\boldsymbol{\Sigma}^{\mathbf{0}}_{\mathbf{1}} \subseteq \boldsymbol{\Sigma}^{\mathbf{0}}_{\mathbf{2}}$). This is no longer true on general cb_0 spaces. This fact is the reason of the small alteration given in the previous definition, which was originally introduced in [Sel06].

Now, to complement the results published in papers such as [Sel08], we give an effective version of this definition using "computable ordinals" (cf. Kleene's notation $(O;<_o)$, recalled in appendix, definition D.0.2). The study of computable ordinals was not in any way a purpose of this internship, and as such the reader can rest assured that no understanding of this notion is required to read the following definitions:

Definition 4.3.2: Lightface Borel Hierarchy

Let $(X, \mathcal{B})$ be an effective cb_0 space. The Lightface Borel hierarchy is defined on natural numbers:

$$\Sigma^0_1(X) = \mathcal{O}_{\text{eff}}(X) \qquad \text{and for any } 2 \leq b < \omega :$$

$$\Sigma^0_b(X) = \left\{ \bigcup_{n\in\mathbb{N}} A_n \setminus A'_n : A_n, A'_n \in \Sigma^0_{b_n}(X) \text{ uniformly and } b_n < b \right\}$$

The Lightface Borel hierarchy is then defined by transfinite induction on the well-founded set $(O, <_o)$. By analogy with the Boldface case, one defines for $b \in O$:

$$\Pi^0_{(b)}(X) = \left\{A^c : A \in \Sigma^0_{(b)}(X)\right\} \qquad \Delta^0_{(b)}(X) = \left\{A : A \in \Sigma^0_{(b)}(X) \cap \Pi^0_{(b)}(X)\right\}$$

The notion of uniformity was again used in the previous definition. In this specific example, $(A^{(j)})_{j\in\mathbb{N}}$ is a uniform family of elements in $\Sigma^0_2(X)$ if there exists a single Turing machine that enumerates the sets $A^{(j)}_i, A'^{(j)}_i \in \Sigma^0_1(X)$ such that, for any $j \in \mathbb{N}$, $A^{(j)} = \bigcup_{i\in\mathbb{N}} A^{(j)}_i \setminus A'^{(j)}_i$.

To familiarize with these definitions, we mention and prove a few elementary properties on the Borel hierarchy that are well-known in Polish spaces (cf. for example [Kec95]):

Property 4.3.3: Inclusions in the Borel hierarchy

1. For any $1 \leq \beta < \omega_1$,
$$\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}} \cup \boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\beta}} \subseteq \boldsymbol{\Delta}^{\mathbf{0}}_{\boldsymbol{\beta+1}}$$
2. For any $b \in O$,
$$\Sigma^0_{(b)} \cup \Pi^0_{(b)} \subseteq \Delta^0_{(b+1)}$$

Along with the following stability properties:

Property 4.3.4: Stability in the Borel hierarchy

1. For any $1 \leq \beta < \omega_1$, $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}$ is stable under countable unions and finite intersections.
2. For any $b \in O$, $\Sigma^0_{(b)}$ is stable under uniform union and finite intersections.

4.3.2 The difference hierarchy

The Borel hierarchy is however not precise enough for our uses: later, we will see (cf. example 7.3.2 and lemma 7.3.3) that it will not always be sufficient to exhibit some phenomenons outside of cb_0 spaces. Between $\boldsymbol{\Sigma}^{\mathbf{0}}_{\mathbf{1}}$ and $\boldsymbol{\Sigma}^{\mathbf{0}}_{\mathbf{2}}$, we will distinguish sets according to the so-called "difference hierarchy" (already introduced in Polish spaces, see [Kec95]). This hierarchy differentiates sets according to the number of set-theoretic operations (union, intersection, complement) one used to build them:

Definition 4.3.5: Boldface difference hierarchy

Let $(X, \mathcal{O}(X))$ be a topological space. The Boldface difference hierarchy is defined by transfinite induction:

- $D_1(\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}) = \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}})$
- $A \in D_{\alpha+1}(\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}})$ if
$$A = U \setminus B$$
where $U \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}$ and $B \in D_\alpha(\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}})$
- For any limit ordinal λ, $A \in D_\lambda(\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}})$ if
$$A = \bigcup_{\gamma<\lambda, \gamma \text{ even}} B_\alpha + 1 \setminus B_\alpha$$
where $(B_\alpha)_{\alpha<\lambda}$ is a growing sequence of sets in $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}$.

One should notice there exists an equivalent definition (which we use in the proofs in appendix):

1. For a growing family of sets $\{A_\gamma\}_{\gamma<\alpha}$, and r the parity function over the ordinals, define:
$$D_\alpha(\{A_\gamma\}_{\gamma<\alpha}) = \bigcup \left\{ A_\gamma \setminus \bigcup_{\theta<\gamma} A_\theta : \gamma < \alpha \text{ and } r(\gamma) \neq r(\alpha) \right\}$$
2. Then for $1 \leq \alpha, \beta < \omega_1$, one has:
$$D_\alpha(\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}) = \left\{ D_\alpha(\{A_\gamma\}_{\gamma<\alpha}) : \{A_\gamma\}_{\gamma<\alpha} \text{ is a growing family of sets in } \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}} \right\}$$

There exists an effectivization of these definitions, mainly studied in papers like [Sel06] and [Sel08]:

Definition 4.3.6: Lightface difference hierarchy

Let $(X, \mathcal{B})$ be an effective cb_0 space. The Boldface difference hierarchy is defined by by induction over O:

- $D_1(\Sigma^0_{(b)}) = \Sigma^0_{(b)})$
- $A \in D_{(a)+1}(\Sigma^0_{(b)})$ if
$$A = U \setminus B$$
where $U \in \Sigma^0_{(b)}$ and $B \in D_{(a)}(\Sigma^0_{(b)})$
- For any name a of a limit ordinal, $A \in D_{(a)}(\Sigma^0_{(b)})$ if
$$A = \bigcup_{(c)<_o(a), |c|_O \text{ even}} B_{(c)+1} \setminus B_{(c)}$$
where $(B_{(c)})_{(c)<_o(a)}$ is a growing sequence of sets in $\Sigma^0_{(b)}$ (we naturally identify such families with $(B_\gamma)_{\gamma<|a|_O}$).

As we did for the Borel hierarchy, we prove a few elementary properties of this new hierarchy. While we do so, we emphasize the similarities between its topological and effective versions:

Property 4.3.7: Inclusions in the difference hierarchy

Boldface hierarchy:

1. For any $1 \leq \alpha < \omega_1$ and $1 \leq \beta < \omega_1$,
$$D_\alpha(\mathbf{\Sigma}^0_\beta) \subseteq D_{\alpha+1}(\mathbf{\Sigma}^0_\beta)$$
2. For any $1 \leq \beta < \omega_1$,
$$\bigcup_{\alpha<\omega_1} D_\alpha\left(\mathbf{\Sigma}^0_\beta\right) \subseteq \mathbf{\Delta}^0_{\beta+1}$$

Lightface hierarchy:

1. For any $a, b \in O$,
$$D_{(a)}(\Sigma^0_{(b)}) \subseteq D_{(a+1)}(\Sigma^0_{(b)})$$
2. For any $b \in O$,
$$\bigcup_{a\in O} D_{(a)}\left(\Sigma^0_{(b)}\right) \subseteq \Delta^0_{(b+1)}$$

For the sake of exhaustiveness, we mention a famous result known as the Haussdorff-Kuratowski theorem (demonstrated in [dB13] - Theorem 70) that gives the converse inclusions in the difference hierarchy:

Theorem 4.3.8: Haussdorff-Kuratowski theorem

Let X be a quasi-Polish space and $1 \leq \beta < \omega_1$. Then:

$$\bigcup_{1\leq\alpha<\omega_1} D_\alpha\left(\mathbf{\Sigma}^0_\beta(X)\right) = \mathbf{\Delta}^0_{\beta+1}(X)$$

Its proof is high above the level of the few elementary properties we mentioned, and its generalization to cb_0 spaces is still an open problem.[2]

Now that all the requirements have been stated, the next sections display the results we have obtained during this internship. Some of them are specified with a mention “original result” in their title: these ones were unpublished by that time. Other titles are specified with an “original proof” mention, which means those ideas were not mine, but their proofs were.

[2]Translation note: It was. Now, if one is interested in such a generalization, they could refer to [CH20], an article Mathieu Hoyrup and I published together in 2020.

5 Results: hierarchies on cb_0 spaces

This section is dedicated to our first original theorem, which asserts that the topological complexity of a set $S \subseteq X$ and its algorithmic complexity (ie. the complexity of its preimage by the standard representation) are identical.

5.1 Context: topological and algorithmic complexity

To study effective topological spaces with representations, we need to be sure that the phenomenons which are captured by representations are exactly the topological phenomenons that happen in the represented spaces. In the following theorems, we call **topological complexity** the height in the difference hierarchy of a set $S \subseteq X$, and **algorithmic complexity** the height in the difference hierarchy of $\delta_X^{-1}(S)$. The following result (published in [dB13], Theorem 68) demonstrates that representations preserve these two notions of complexity:

Theorem 5.1.1: Equivalence of topological and algorithmic complexity

Let $(X, \mathcal{B})$ be a cb_0 space. For any $1 \leq \alpha, \beta < \omega_1$ and $S \subseteq X$:

$$S \in D_\alpha \left(\boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}(X) \right) \iff \delta_X^{-1}(S) \in D_\alpha \left(\boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}(\text{dom}(\delta_X)) \right)$$

As $D_1(\boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}) = \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}$, we obtain the corollary:

$$S \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}(X) \iff \delta_X^{-1}(S) \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}(\text{dom}(\delta_X))$$

5.2 Result: effective equivalence of topological and algorithmic complexity

One of the main goals of this internship was to find an effective version of the previous theorem (ie. with the Lightface hierarchies instead of the Boldface hierarchies). The case Π^0_2 (and so Σ^0_2) had already been tackled in [HRSS19], but the behavior of the representation was unknown in the difference hierarchy, and in the Borel hierarchy for ordinals greater than 2.

The first result that I obtained during this internship answered this question. I demonstrated the following effective analogous version of the previous theorem 5.1.1:

[Original result] - Theorem 5.2.1: Effective equivalence of topological and algorithmic complexities

Let $(X, \mathcal{B})$ be an effective cb_0 space. For any $a, b \in O$ and $S \subseteq X$:

$$S \in D_{(a)} \left(\Sigma^0_{(b)}(X) \right) \iff \delta_X^{-1}(S) \in D_{(a)} \left(\Sigma^0_{(b)}(\text{dom}(\delta_X)) \right)$$

As with the previous theorem, we obtain the following effective corollary:

$$S \in \Sigma^0_{(b)}(X) \iff \delta_X^{-1}(S) \in \Sigma^0_{(b)}(\text{dom}(\delta_X))$$

Such a result highlights that cb_0 spaces are a context where computability is both possible and desirable, as the topological behavior of sets will be exactly the one we wished for.

5.3 Proof of the result

The complete proof can be found in Appendix C.1. It is mainly inspired by the one published in [dB13] for its Boldface equivalent, which is itself based on a transformation introduced in [Ray07]:

Transformation $B : A \subseteq \mathcal{N} \mapsto B(A) \subseteq X$

Let $(X, \mathcal{B})$ be a cb_0 space and $A \subseteq \mathcal{N}$ be a subset of $\text{dom}(\delta_X)$. We define:

$$B(A) \triangleq \{x \in X : \delta_X^{-1}(x) \cap A \text{ is non-meager in } \delta_X^{-1}(x)\}$$

The proof of Lemma 17 in [Ray07] assures that A and $B(A)$ have the same hierarchical complexities in the Borel hierarchy. In formal words, if $A \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}$, then $B(A) \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}$. This is the key for the proof of theorem 5.1.1 (or Theorem 68 in [dB13]), because this transformation has the following property: if $S \subseteq X$ is a Borel set, then $B(\delta_X^{-1}(S)) = S$.

What is really at stake here is to find an effective version of the previous lemma. Proceeding similarly as in [Ray07], We demonstrate that if A is an effective set, then $B(A)$ is an effective set of the same topological complexity. Additionally, uniformity is preserved by this transformation. Thus, we obtain the following lemma:

[Original result] - Lemma 5.3.1: Modified Lemma 17 of [Ray07]

Let $(X, \mathcal{B})$ be an effective cb_0 space. For any $b \in O$ and $A \subseteq \text{dom}(\delta_X)$,

$$A \in \Sigma^0_{(b)}(\text{dom}(\delta_X)) \implies B(A) \in \Sigma^0_{(b)}(X)$$

Additionally, the transformation preserves uniformity for families of subsets in $\text{dom}(\delta_X)$.

The idea and the proof are indeed mine, but I have to underline that both are slight effectivizations of results already published in [Ray07]. So that this proof is more a bibliographic research than a mathematical novelty. That being said, with this lemma, it is relatively easy to end the proof of theorem 5.2.1:

$\implies$: This implications follows from the effective continuity of δ_X.

$\impliedby$: Let me prove now the converse implication. If $\delta_X^{-1}(S) \in D_{(a)}(\Sigma^0_{(b)}(\text{dom}(\delta)))$, we write: $\delta_X^{-1}(S) = D_{(a)}(\{A_c\}_{c <_o a})$. And in a very similar way to [dB13], we show that S can be written as $S = D_{(a)}(\{B(A_c)\}_{c <_o a})$. Thus $S \in D_{(a)}(\Sigma^0_{(b)}(X))$ by applying lemma 5.3.1.

This concludes our investigation in cb_0 spaces as represented topological spaces. Of course, numerous questions have yet to find their answers, and a book unifying all these theories (like [Kec95] did for Polish spaces) has yet to be published as far as we know. We do think, however, that such a synthesis work would be extremely profitable.

For the time being, this thesis will continue onto other topics. In the following section, we will cover a more concrete facet of computability on cb_0 spaces, by exploring a class of maps we call "piecewise-computable".

6 Results: piecewise-computability

6.1 Context: introducing the notion

One of the most common criticisms on Type-2 computability is that computable maps are necessarily continuous (property 3.2.4): for example, a function as simple as $f : x \in \mathbb{R} \mapsto \lfloor x \rfloor \in \mathbb{R}$ is not computable, which is why several mathematicians seriously question the interest of an effective analysis based on such models.

[Zie12] claims those criticisms are not relevant, as many non-continuous maps are "computable" if discrete information is added to its arguments. In practical computer science, one often adds several parameters to a function f, which qualify its main input variable: for example, the floor function becomes continuous if one has a boolean telling whether the input variable is an integer or not. In this section, we focus on this class of maps, that we call "piecewise-computable".

Definition 6.1.1: Piecewise-computability

Let X and Y be two effective cb_0 spaces. A map $f :\subseteq X \mapsto Y$ is **piecewise-computable** if there exists a cover $P = \{P_i\}_{i\in I}$ of $\mathrm{dom}(f)$ such that for any $i \in I$, $f_{|P_i}$ is computable.

During this internship, we have asked the following question: if a new bit of information is necessary to compute a map, is this new piece of information inherent to the points of X rather than to their names? To rephrase this, is this always possible to associate a cover of $\mathcal{N}$ to a cover of X related to a piecewise-computable map? And reciprocally? Additionally, what are their topological complexity?

Here is an outline of the results developed in the subsections to come:

- *[Original result]*: It is equivalent for a map to be piecewise-computable with a countable cover, and for its realization to be piecewise-computable in the Baire space. The proof uses the transformation $A \mapsto B(A)$ we introduced in the previous section.
- It is equivalent for a map to be piecewise-computable with a finite cover of cardinal n on X, and for its realization to be piecewise-computable with the same cardinality on the Baire space, if the complexity of the elements of this cover is Σ^0_1 or Π^0_1.
- *[Original result]*: For higher complexities, provided that the realization of a map f has a finite cover in the space of names, then there is a countable cover of X that makes f piecewise-computable. But there exists a counter-example showing that the realization of a map can be piecewise-computable for a finite cover on the Baire space, while the associated map has no cover of the same cardinal on X.

6.2 Results: the countable case

During this internship, we have obtained the following original result in the context of countable covers:

[Original result] - Theorem 6.2.1: Piecewise-computability: countable cover

Let $(X, \mathcal{B}^{(X)})$ and $(Y, \mathcal{B}^{(Y)})$ be two cb_0 topological spaces and $f :\subseteq X \mapsto Y$ be a map. The two following assertions are equivalent:

1. There exists a cover $\{P_i\}_{i\in\mathbb{N}}$ of $\mathrm{dom}(\delta_X)$ of complexity $\Sigma^0_{(b)}(\mathcal{N})$ and a family of computable maps $\{F_i\}_{i\in\mathbb{N}}$ such that:
$$\forall i \in \mathbb{N}, \forall p \in \mathrm{dom}(F) \cap P_i, \quad \delta_Y \circ F_i(p) = f \circ \delta_X(p)$$
2. There exists a cover $(Q_i)_{i\in\mathbb{N}}$ of X of complexity $\Sigma^0_{(b)}(X)$ such that for any $i \in \mathbb{N}$, $f_{|Q_i}$ is a computable map.

Here is a short ovierview of the proof, which complete redaction is of course available in appendix C.2.2. We can easily see that if a map $f : X \mapsto Y$ is piecewise-computable for the cover $(Q_i)_{i\in I}$ of X, and if $P_i = \delta_X^{-1}(Q_i)$, then the realization of f on Baire is piecewise-computable for the cover $(P_i)_{i\in I}$. Such a thing is also true in the finite case. In other words, **to any cover of X related to a piecewise-computable map $f :\subseteq X \mapsto Y$, one can associate a cover of $\mathcal{N}$ of same cardinality and complexity.**

The converse implication is more interesting, as it uses the transformation $A \mapsto B(A)$ and some properties on computable functions with dense domain (in a convenient set). The key idea is that provided with an oracle enumerating the cylinders covering the domain of a constant map $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$, we can compute the related constant without really having to compute a point in the domain.

6.3 Results: the finite case

In the finite case, depending on the complexity of the cover, two possibilities arise. The first was already known, the second is an original result of this paper.

1st case: For a cover $\{P_1, P_2\}$ of $\mathrm{dom}(f)$ such that $P_1 \in \Sigma_1^0(\mathrm{dom}(f))$.

Theorem 6.3.1: Piecewise-computability : finite cover of Σ_1^0 and Π_1^0 sets

With the same notations, the two following assertions are equivalent:

1. There exists a cover $\{P_1, P_2 = {P_1}^c\}$ of $\mathrm{dom}(f)$, where $P_1 \in \Sigma_1^0(\mathcal{N})$, and two computable maps $F_i :\subseteq \mathcal{N} \mapsto \mathcal{N}$ such that:
$$\forall i \in \{1,2\}, \forall p \in \mathrm{dom}(F) \cap P_i, \quad \delta_Y \circ F_i(p) = f \circ \delta_X(p)$$
2. There exists a cover $\{Q_1, Q_2\}$ of X, where the complexity of Q_i is the one of P_i, and such that $f_{|Q_1}$ and $f_{|Q_2}$ are computable.

The proof is specific to a cover made of open sets, as it intrinsically depends on the fact that P_1 is an effective open set of $\mathrm{dom}(f)$, and that the standard representation is effectively open. However, using the proof of the countable case, the following compromise can be obtained:

2nd case: For a cover $\{P_1, P_2 = {P_1}^c\}$ of $\mathrm{dom}(f)$ such that $P_1 \in \Sigma_{(b)}^0(\mathrm{dom}(f))$, where $|b|_O \geq 2$.

[Original result] - Theorem 6.3.2: Piecewise-computability : finite cover of $\Sigma_{(b)}^0$ sets, where $|b|_O \geq 2$

With the same notations,

Suppose there exists a cover $\{P_1, P_2 = {P_1}^c\}$ of $\mathrm{dom}(f)$, where $P_1 \in \Sigma_{(b)}^0(\mathcal{N})$, and two computable maps $F_i :\subseteq \mathcal{N} \mapsto \mathcal{N}$ such that:
$$\forall i \in \{1,2\}, \forall p \in \mathrm{dom}(F) \cap P_i, \quad \delta_Y \circ F_i(p) = f \circ \delta_X(p)$$
Then there exists a countable cover $\{Q_i\}_{i\in\mathbb{N}}$ of X, where $Q_i \in \Sigma_{(b)}^0(X)$, such that for any $i \in \mathbb{N}$, $f_{|Q_i}$ is a computable map.

So we have an equivalence for countable covers and a result of finite-countable form, but what could be said about a finite-finite case? Having a cover of two elements of the Baire space is not always sufficient to obtain a cover of same cardinality of X. This is the topic of the following counter-example originally published in [Zie12], for which we provide a new proof correcting a mistake for admissible representations.

Example 6.3.3: Piecewise-computability : counter-example

Define $f : [0,1] \mapsto \mathbb{R}$ the map such that $f(x) = x \quad \text{if } x \in \mathbb{Q} \cap [0,1)$
$f(x) = x - 1 \text{ if } x \in ([0,1] \setminus \mathbb{R}) \cup \{1\}$

We map the segment $[0,1]$ to the circle $\mathcal{S}^1$ with the following map: $q : x \mapsto x \mod 1$. There exists a map $\tilde{f} : \mathcal{S}^1 \mapsto \mathbb{R}$ such that $\tilde{f} \circ q = f$. And:

1. There exists computable maps $F_i :\subseteq \mathcal{N} \mapsto \mathcal{N}$ $(i = 1, 2)$ and a set $A \in \Sigma^0_3$ such that, by defining $P_1 = A$ and $P_2 = A^c$:

$$\forall i \in \{1,2\}, \forall p \in P_i \cap \text{dom}(\delta_X), \quad \delta_{\mathbb{R}} \circ F_i(p) = \tilde{f} \circ \delta_{\mathcal{S}^1}(p)$$

2. But there is no partition $\{B_1, B_2\}$ of $\mathcal{S}^1$ that would make f continuous on each of its elements; and *a fortiori* computable.

One of the ideas that came during this internship was to generalize this example to any cardinality $k \in \mathbb{N}$ of the cover. Indeed, rather than considering rational and irrational numbers of $\mathcal{S}^1$, a cover of cardinal k (for any k) can be obtained using similar ideas. We consider the $k-1$ first prime numbers and their p-adic numbers (rational numbers whose denominator in irreducible form is the power of a prime number p). They yield $k-1$ dense sets (and whose complements are also dense in $\mathcal{S}^1$). Then we complete the cover with the complement of the union of those $k-1$ dense sets. This process leads to the following generalized counter-example:

[Original result] - Example 6.3.4: Piecewise-computability: generalized counter-example

Let $k \in \mathbb{N}, k \geq 2$ and let $p_1, ..., p_{k-1}$ be the first $k-1$ prime numbers. Let $\mathbb{Q}_p$ be the set of rational number whose irreducible form $\frac{n}{d}$ is such that d is a power of p. Define now $f : [0,1] \mapsto \mathbb{R}$ as the following map:

$$\begin{aligned}
f(x) &= x && \text{if } x \in \mathbb{Q}_{p_1} \cap [0,1) \\
f(x) &= x - 1/k && \text{if } x \in \mathbb{Q}_{p_2} \cap (0,1) \cup \{1\} \\
&\dots \\
f(x) &= x - (k-1)/k && \text{if } x \in \mathbb{Q}_{p_{k-1}} \cap (0,1) \\
f(x) &= x - 1 && \text{if } x \in ([0,1] \setminus (\mathbb{Q}_{p_1} \cup ... \cup (\mathbb{Q}_{p_{k-1}}))
\end{aligned}$$

We map the segment $[0,1]$ to the circle $\mathcal{S}^1$ with the following map: $q : x \mapsto x \mod 1$. There exists a map $\tilde{f} : \mathcal{S}^1 \mapsto \mathbb{R}$ such that $\tilde{f} \circ q = f$. And:

1. There exists k computable maps $F_i :\subseteq \mathcal{N} \mapsto \mathcal{N}$ $(i = 1, ..., k)$ and a partition into k subsets $P_1, ..., P_k \in \Sigma^0_3$ such that:

$$\forall i \in \{1, ..., k\}, \forall p \in P_i \cap \text{dom}(\delta_X), \quad \delta_{\mathbb{R}} \circ F_i(p) = \tilde{f} \circ \delta_{\mathcal{S}^1}(p)$$

2. But there is no partition $\{B_1, ..., B_k\}$ of $\mathcal{S}^1$ that would make f continuous on each of its elements; and *a fortiori*, computable.

This concludes our exploration of the different subcases we studied for piecewise-computability and piecewise-realization. However, knowing whether a finite cover of X can be associated to a finite cover of the Baire space is still an open question. Indeed, we showed it wasn't possible for covers of same cardinality, but there may be cases where the cardinality could grow and remain finite.

7 Results: non-countably-based spaces, the example of real polynomials

The end of this internship has been the occasion to investigate some T_0 spaces that do not have a countable base, with a focus on the example of the set of real polynomials $\mathbb{R}[X]$. We wondered what were the topological characteristics of this space, and whether a representation could enable us to manipulate its elements. Our approach followed those two very interlinked directions:

1. The algorithmic hierarchies seemed to differ from the topological ones: some sets were written as differences of open sets in $\mathcal{N}$, without us being able to write them as something "less than" a $\boldsymbol{\Delta}^0_2$ set in X.
2. Some usual theorems in Descriptive Set Theory (which were, *a priori*, quite distant from our main preoccupations on represented spaces), like the Hausdorff-Kuratowski or the Wadge theorems, seemed to no longer apply on $\mathbb{R}[X]$.

First, here is an informal summary of the results we obtained on this topic:

1. We found two topologically $\boldsymbol{\Delta}^0_2$-complete sets that do belong at inferior levels of the difference hierarchy algorithmically (ie. in the space of names). They prove that the topological and algorithmic complexities do no coincide on $\mathbb{R}[X]$.
2. To obtain those results, we created with Mathieu Hoyrup new methods and tools to prove the $\boldsymbol{\Delta}^0_2$-hardness of a set. Mathieu created the method, but its redaction and proofs here are mine.
3. We eventually identified these two sets as counter-examples to the aforementioned theorems.

The content of this section displays a more rigorous yet pedagogical development. For the sake of clarity, all the additionnal technical details and reflections can be found in Appendix C.3.

7.1 Definitions and topology on the set of real polynomials

We first define $\mathbb{R}[X]$ as the following topological space:

Definition 7.1.1: Definition of $\mathbb{R}[X]$ and its topology

Define $\mathbb{R}[X]$ as the inductive limit of the sets $\mathbb{R}_n[X]$, equipped with the associated **coPolish topology**, ie:

$$\mathbb{R}[X] = \bigcup_{n\geq 0} R_n[X] \quad \text{and} \quad \mathcal{O}(\mathbb{R}[X]) = \{O \subseteq \mathbb{R}[\mathbb{X}] : \forall n \in \mathbb{N}, O \cap \mathbb{R}_n[X] \in \mathcal{O}(\mathbb{R}_n[X])\}$$

How can we represent $\mathbb{R}[X]$? We first use the Cauchy representation $\delta_{\mathrm{C}}^{n+1}$ for the spaces $\mathbb{R}_n[X]$ (which are Polish, the construction can be read in section C.3.1). We then construct the following representation in the ways of [KS05]:

Definition 7.1.2: A representation for $\mathbb{R}[X]$

Define $\delta_{\mathcal{P}} :\subseteq \mathcal{N} \mapsto \mathcal{N}$ by:

$$\delta_{\mathcal{P}}(n \cdot p) = x \iff x \in \mathbb{R}_n[X] \text{ and } \delta_C^{n+1}(p) = x$$

Why do we use this representation? Because is general enough, thanks to its two following properties:

Property 7.1.3: Representation $\delta_{\mathcal{P}}$

1. $\delta_{\mathcal{P}}$ is an admissible representation of $\mathbb{R}[X]$.
2. The final topology wrt. $\delta_{\mathcal{P}}$ is the coPolish topology.

Now that the space $\mathbb{R}[X]$ is defined as a represented space, we can start studying its topology. First, we notice that it is not metrizable (cf. property C.3.5, demonstrated in appendix C). It is not even countably-based, because of the following characterization of its open sets:

Property 7.1.4: A basis of open sets for $\mathbb{R}[X]$

The coPolish topology on $\mathbb{R}[X]$ has the following basis:

- The open sets of the product topology.
- Along with the sets defined by, for any $h : \mathbb{N} \mapsto \mathbb{R}_+^*$:

$$O_h = \left\{ P = \sum_{k=0}^{\deg(P)} p_k X^k \in \mathbb{R}[X] : \forall j, p_j < h(j) \right\}$$

7.2 Γ-hardness

7.2.1 General considerations, new definitions and their motivations

In this subsection, Γ (or $\Gamma(X)/\Gamma(\tau)$ if the underlying space/topology is not obvious) is a complexity class in the difference hierarchy.

We now develop a method proving that a set S does *not* belong in a complexity class Γ. A brief bibliographic search shows that in Polish spaces, [Kec95] (Chapter 22) defines the following notion of Γ-hardness, which directly generalize it to cb_0 spaces:

Definition 7.2.1: Γ-hardness on a cb_0 space

A subset S of a cb_0 space X is **Γ-hard** if for any $A \in \Gamma(\mathcal{N})$, there exists a total map $f : \mathcal{N} \mapsto X$ such that $f^{-1}(S) = A$.

One out of the many interests of this notion resides in the following property (with $\tilde{\Gamma}$ the dual class of the complexity class Γ, $\tilde{\Gamma}$ being defined as $\tilde{\Gamma} = \{S^c : S \in \Gamma\}$):

Property 7.2.2: $\tilde{\Gamma}$-hardness $\implies S \notin \Gamma$

Let X be a topological space, Γ a complexity class such that $\Gamma \neq \tilde{\Gamma}$, and $S \subseteq X$. Then:

$$S \text{ is } \tilde{\Gamma}\text{-hard} \implies S \notin \Gamma$$

An interesting fact is that this property becomes an equivalence in the context of quasi-Polish spaces. This is the Wadge theorem, written in [Kec95] in the context of Polish spaces (Theorem 22.10), and which was extended to quasi-Polish spaces by Theorem 48 of [dB13].[3]

However, such a method can't apply to the space $\mathbb{R}[X]$, where reductions only capture the complexity of the preimages $\delta_{\mathcal{P}}^{-1}(S)$ rather than the complexity of the sets $S \subseteq \mathbb{R}[X]$ (cf. property C.3.6). Because of that, reductions overlook any difference between topological and algorithmic complexities, even though we are looking for such a difference.

In view of this failure, we need a new method showing that $S \notin \Gamma$. Mathieu Hoyrup suggested to pay attention to this elementary lemma:

[3] Note added during translation: in [CH20], we showed an extension of this result to cb_0 spaces.

Lemma 7.2.3: Belonging in a countable class

Let (X, τ) be a topological space and Γ be a class of the Borel or the difference hierarchy. For any $1 \leq \alpha, \beta < \omega_1$ and $S \subseteq X$:

$$S \in \Gamma(\tau) \iff \exists \tau' \subseteq \tau \text{ countably-based, } S \in \Gamma(\tau')$$

And with this lemma, the problem of hardness in any topological space was reduced to cb_0 spaces:

Definition 7.2.4: Γ-hardness* in topological spaces

Let (X, τ) be a topological space and $S \subseteq X$. S is **Γ-hard*** (ie. $\Gamma(\tau)$-hard*) if for every countably-based topology $\tau' \subseteq \tau$, S is $\Gamma(\tau')$-hard.

By property C.3.7, we obtain that if S is $\tilde{\Gamma}$-hard* (and that $\tilde{\Gamma} \neq \Gamma$), then $S \notin \Gamma$. And in our particular case (where we focus on $\boldsymbol{\Delta}^0_2$-hardness*), we demonstrate (property C.3.8) that if S is $\boldsymbol{\Delta}^0_2$-hard*, then for any $\alpha < \omega_1$, $S \notin D_\alpha(\boldsymbol{\Sigma}^0_1)$.

7.2.2 Method: proving $\boldsymbol{\Delta}^0_2$-hardness

In this subsection, we provide a method in order to demonstrate that some sets $S \subseteq \mathbb{R}[X]$ are $\boldsymbol{\Delta}^0_2$-hard*, which we will then apply in the following developments.

This subsection is optional for the global understanding of this paper, as it only details technical aspects of reductions. The reader may skip this part and go directly to the next subsection to discover the results.

As for any reduction in traditional complexity theory, we introduce a standard $\boldsymbol{\Delta}^0_2$-hard* problem: the "asymptotic" decision of the limit value of a converging sequence of bits 0 or 1.

Property 7.2.5: Converging sequences of $\{0,1\}^{\mathbb{N}}$

Let $\{0,1\}^{\mathbb{N}}_{\text{CV}}$ be the set of converging sequences with values in $\{0,1\}$. A set $A \subseteq \mathcal{N}$ is an element of $\boldsymbol{\Delta}^0_2(\mathcal{N})$ if and only if there exists a continuous map $f_A : \mathcal{N} \mapsto \{0,1\}^{\mathbb{N}}_{\text{CV}}$ such that :

$$\lim_{n \to +\infty} f_A(p)_n = 1 \iff p \in A$$

With this property, any set that is reducible to $\{0,1\}^{\mathbb{N}}_{\text{CV}}$ is $\boldsymbol{\Delta}^0_2$-hard*; but we have yet to show what could be such a reduction. This is the topic of the following lemma:

[Original proof] - Lemma 7.2.6: A method of reduction for $\boldsymbol{\Delta}^0_2$

Let $S \subseteq \mathbb{R}[X]$.
Suppose that for any countable-based topology $\tau' \subseteq \tau$ (whose base is made of the product topology and of a countable family of functions $\{h_i\}_{i \in \mathbb{N}}$),
if you define $Y = \{P = \sum_{k \geq 0} p_k X^k \in \mathbb{R}[X] : \forall i, p_i < h_0(i), ..., h_i(i)\}$,
there exists a continuous map (for the product topology) $f_{S,Y} : \{0,1\}^{\mathbb{N}}_{\text{CV}} \mapsto Y$ such that:

$$f_{S,Y}(u) \in S \iff u \text{ converges towards } 1$$

Then S is $\boldsymbol{\Delta}^0_2$-hard*.

This lemma gives us an algorithmic procedure in order to prove that a subset $S \subseteq \mathbb{R}[X]$ is $\mathbf{\Delta^0_2}$-hard*. First, let $h_i : \mathbb{N} \mapsto \mathbb{R}^*_+$ be a countable family of functions ($i \in \mathbb{N}$) and let u be an element of $\{0,1\}^{\mathbb{N}}_{\mathrm{CV}}$. Then:

1. At each step n, we enumerate the n first coefficients of a polynomial P with precision $1/2^n$. This approximating polynomial at time n is called P_n. (This ensures the continuity of the procedure according to the product topology on Y).
2. By maintaining inequalities of the coefficients according to the functions h_i, we ensure the polynomials P_n to be in Y.
3. At each value change of the sequence u of bits, we add coefficients which remove (if the change was $1 \to 0$) or put back (if $0 \to 1$) the new polynomial P_n at step n.
4. Then the sequence P_n converges towards a polynomial $P \in Y$ such that $P \in S$ iff u converges towards 1. And if the whole procedure comes to an end, then the set S is $\mathbf{\Delta^0_2}$-hard* by the previous lemma.

While this method proves that a set is $\mathbf{\Delta^0_2}$-hard*, does this method capture every $\mathbf{\Delta^0_2}$-hard* sets? Is every $\mathbf{\Delta^0_2}$-hard* subset of $\mathbb{R}[X]$ reducible to the previous problem? This question is still open for now.[4]

7.3 Results: mismatch of topological and algorithmic complexities on $\mathbb{R}[X]$

With new notions in our toolbox, we can introduce the final results of this internship. This subsections focuses on two sets that are $\mathbf{\Delta^0_2}$-complete* (topological complexity), but whose preimage under $\delta_{\mathcal{P}}$ (ie. algorithmic complexity) belongs at inferior levels of the difference hierarchy.

The idea to focus on polynomial of even degree is directly inspired by an advice from Mathieu Hoyrup, as this is a set whose $\mathbf{\Delta^0_2}$-hardness* is quite easy to prove. This leads to the following example:

[Original result] - Example 7.3.1: A first $\mathbf{\Delta^0_2}$-complete* set S_1

Define:

$$S_1 = \{P \in \mathbb{R}[X] : \deg(P) \text{ is even } \}$$

Then S_1 is $\mathbf{\Delta^0_2}$-hard*, but $\delta_{\mathcal{P}}^{-1}(S_1) \in D_\omega\left(\mathbf{\Sigma^0_1}(\mathrm{dom}(\delta_{\mathcal{P}}))\right)$.

While the results we were looking for are already consequences of this example, we have obtained another set with similar properties, but at the lowest level of the difference hierarchy. We highlight that its structure is not without being reminiscent of the proof of non-metrizability for $\mathbb{R}[X]$. This set is given in the following example, which confirms the intuitions we had when we started this section:

[Original result] - Example 7.3.2: A second $\mathbf{\Delta^0_2}$-complete* set S_2

Define:

$$S_2 = \left\{ P = \sum_{j=0}^{n-1} \frac{1}{k_j} X^{d_j} + \frac{1}{k_n} X^{d_n} : k_j, d_j \in \mathbb{N}, \forall j, k_j \geq 2k_{j-1} \text{ and } k_{n-1} = d_n \right\}$$

Then S_2 is $\mathbf{\Delta^0_2}$-hard*, but $\delta_{\mathcal{P}}^{-1}(S_2) \in D_2\left(\mathbf{\Sigma^0_1}(\mathrm{dom}(\delta_{\mathcal{P}}))\right)$.

Those two examples show that there are two different notions of complexity on $\mathbb{R}[X]$: the algorithmic complexity (of $\delta^{-1}(S) \subseteq \mathrm{dom}\,\delta_{\mathcal{P}}$) and topological complexity (of $S \subseteq \mathbb{R}[X]$). And contrary to the setting of cb_0 spaces (proved in [dB13]), they are not equivalent here.

[4] Note added during translation: we later proved in [CH20] this is the case in cb_0 spaces, but the general case is still open.

Why did we introduced the differency hierarchy in the first place? Indeed, the previous statement is all the more interesting that it does not happen on the Borel hierarchy. This is what we prove in this (already known) statement:

Lemma 7.3.3: Borel topological and algorithmic complexities coincide

Consider $\mathbb{R}[X]$ equipped with the admissible representation $\delta_{\mathcal{P}}$. For any $S \subseteq \mathbb{R}[X]$ and $1 \leq \beta < \omega_1$,

$$S \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}(\mathbb{R}[X]) \iff \delta_{\mathcal{P}}^{-1}(S) \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}(\mathrm{dom}(\delta_{\mathcal{P}}))$$

This theorem, which is similary to theorem 5.2.1, is actually far less profound, as its proof only consists in working with the definition of the coPolish topology.

We believe examples 7.3.1 and 7.3.2 raise several extremely interesting questions. In countably-based spaces, the topological and computational point of view were equivalent. In $\mathbb{R}[X]$, this is no longer true, and that questions our knowledge of representations: some topological information is lost by the representation. In order to deepen our understanding of the reasons behind this mismatch on $\mathbb{R}[X]$, we ask the following question: what phenomenons can be captured by this theory of represented spaces and by our notion of computability? For example, a characterization of spaces where the equivalence is true could shed a new light on the theory of represented spaces itself.

Mathieu Hoyrup suggests the following intuition: this mismatch between the two hierarchies lies in the difference between sequential and topological behaviors on $\mathbb{R}[X]$. For example, this space is not "Fréchet-Urysohn": topological closure and sequential closure can be different. However, this has yet to be formalized into a precise mathematical statement, and we think that this open question is interesting enough to conclude this development.

7.4 Results: counter-examples to usual theorems in Descriptive Set Theory

Now that the important results about $\mathbb{R}[X]$ were introduced in the previous subsection, we now explore two topological consequences of the examples we found earlier: the Hausdorff-Kuratowski and the Wadge theorems, well-known in Descriptive Set Theory, no longer apply on $\mathbb{R}[X]$.

The Hausdorff-Kuratowski theorem

We recall the Hausdorff-Kuratowski theorem in the context of quasi-Polish spaces:

Theorem 4.3.8: Haussdorff-Kuratowski theorem

Let X be a quasi-Polish space and $1 \leq \beta < \omega_1$. Then:

$$\bigcup_{1 \leq \alpha < \omega_1} D_\alpha\left(\boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}(X)\right) = \boldsymbol{\Delta}^0_{\boldsymbol{\beta+1}}(X)$$

The two sets S_1 and S_2 imply that this theorem is not longer true on $\mathbb{R}[X]$:

[Original result] - Theorem 7.4.1: Counter-example to the Hausdorff-Kuratowski theorem

Consider $\mathbb{R}[X]$ equipped with the coPolish topology. There exists a set $S \in \boldsymbol{\Delta}^0_{\mathbf{2}}(\mathbb{R}[X])$ such that:

$$\forall \alpha < \omega_1, S \notin D_\alpha(\boldsymbol{\Sigma}^0_{\mathbf{1}}(\mathbb{R}[X]))$$

The Wadge theorem

Likewise, the Wadge theorem was already enunciated in the context of quasi-Polish spaces:

Theorem 7.4.2: Wadge Theorem

Let X be a quasi-Polish space and Γ a complexity class in the difference/Borel hierarchy such that $\Gamma \neq \tilde{\Gamma}$. Then:

$$S \notin \Gamma(X) \iff S \text{ is } \tilde{\Gamma}\text{-hard}$$

And likewise, the set S_2 implies that this theorem is no longer true on $\mathbb{R}[X]$:

[Original result] - Theorem 7.4.3: Counter-example to the Wadge theorem

Consider $\mathbb{R}[X]$ equipped with the coPolish topology. There exists a set $S \subseteq \mathbb{R}[X]$ such that $S \notin D_2(\boldsymbol{\Sigma}_1^0(\mathbb{R}[X]))$, but S is not $\tilde{D}_2(\boldsymbol{\Sigma}_1^0)$-hard (in the usual sense).

We think that these two consequences hold a double interest. The first is topological, and demonstrates that there is no correspondence between computability and topology outside of countably-based spaces. The two counter-examples are an illustration, amongst many others, that little is known about the theory of represented spaces outside of cb_0 spaces, and that our understanding of the phenomenons that happen outside of cb_0 spaces is still incomplete.

In my opinion, the second interest lies in the mathematical approach. Indeed, these two counter-examples are fundamentally topological: they are results which were successfully abstracted from any computability context, and which only focus on the topological complexity of a set. They are only of a topological nature. This illustrates the generality of this approach in a very interesting way, and epitomizes the fruitfulness of the theory of represented spaces in relationship with other fields of knowledge. We were able to obtain new results in a classical mathematical area such as topology, and we hope that our study successfully illustrates the usefulness of extending computability theory outside of its traditional setting on natural numbers.

8 Conclusion

We first demonstrated that the theory of effective represented spaces, well-known on Polish spaces and studied on quasi-Polish spaces, can be extended in the context of countably-based spaces: we indeed showed that computability perfectly matches topological behaviors on cb_0 spaces. We see these results as encouraging. While extending the whole Descriptive Set Theory to countably-based spaces would require many efforts, such an initiative would be relevant and worthwhile.

We also focused on the inherent limits of continuity induced by computability, and hope we have demonstrated they are not as restrictive as the impression we first had. Furthermore, the counter-examples show that the use of a representation brings additional information, and can reduce the cardinality of a partition associated to a piecewise-computable map. As a result, computability on the Baire space bears more information than the topology itself, which is, in our opinion, another evidence of the interests held by this notion of computability.

Of course, the search of an ideal mathematical context for computability is far from over. The several examples we have found on the space of real polynomials $\mathbb{R}[X]$ highlight the fact that our understanding of the theory of represented spaces is still very limited. We have indeed showed that the hierarchies induced by the topology and by the representation are different on $\mathbb{R}[X]$, and we have yet to understand the mathematical reasons of such a mismatch. We think (with all necessary precautions) that for real polynomials, this is related to the sequential aspects of the topology (for example, the sequential closure is not always equal to the topological one, ie. the coPolish space $\mathbb{R}[X]$ is not Fréchet-Urysohn), but there is still a need for a mathematical statement formalizing this intuition. These problems remain open for the time being, and a mathematical characterization of the topological behaviors that can be expressed by a representation has yet to be found. I think studying some other coPolish spaces and examples could provide us with an hypothesis to solve those questions.

In the end, beyond these very specific outcomes, we hope that we have sucessfully convinced you that the links between computability and topology are stronger than what you may have thought *a priori*. They can provide insights and results to each other, as shown with our development of real polynomials. Even if we still know little about the crossing of these fields of study, this paper proves that it can lead to fruitful collaborations.

Bibliography

[BH02] Vasco Brattka and Peter Hertling. Topological properties of real number representations. *Theoretical Computer Science*, 284(2):241–257, 2002.

[CH20] Antonin Callard and Mathieu Hoyrup. Descriptive complexity on non-polish spaces. In *37th International Symposium on Theoretical Aspects of Computer Science, STACS 2020, March 10-13, 2020, Montpellier, France*, volume 154 of *LIPIcs*, pages 8:1–8:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

[dB13] Matthew de Brecht. Quasi-polish spaces. *Annals of Pure and Applied Logic*, 164(3):356–381, 2013.

[HRSS19] Mathieu Hoyrup, Cristóbal Rojas, Victor Selivanov, and Donald M. Stull. Computability on quasi-polish spaces. In *Descriptional Complexity of Formal Systems*, pages 171–183. Springer International Publishing, 2019.

[Kec95] A. Kechris. *Classical Descriptive Set Theory*. Graduate Texts in Mathematics. Springer New York, 1995.

[KS05] Daren Kunkle and Matthias Schröder. Some examples of non-metrizable spaces allowing a simple type-2 complexity theory. *Electronic Notes in Theoretical Computer Science*, 120:111 – 123, 2005.

[KW85] Christoph Kreitz and Klaus Weihrauch. Theory of representations. *Theoretical Computer Science*, 38:35 – 53, 1985.

[Ray07] Jean Saint Raymond. Preservation of the borel class under countable-compact-covering mappings. *Topology and its Applications*, 154(8):1714 – 1725, 2007.

[Sch02] Matthias Schröder. Extended admissibility. *Theoretical Computer Science*, 284(2):519 – 538, 2002.

[Sel06] Victor L. Selivanov. Towards a descriptive set theory for domain-like structures. *Theoretical Computer Science*, 365(3):258 – 282, 2006. Spatial Representation: Discrete vs. Continuous Computational Models.

[Sel08] Victor Selivanov. On the difference hierarchy in countably based t0-spaces. *Electronic Notes in Theoretical Computer Science*, 221:257 – 269, 2008.

[Wei85] Klaus Weihrauch. Type 2 recursion theory. *Theoretical Computer Science*, 38:17 – 33, 1985.

[Wei00] Klaus Weihrauch. *Computable Analysis - An Introduction*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2000.

[Zie12] Martin Ziegler. Real computation with least discrete advice: A complexity theory of nonuniform computability with applications to effective linear algebra. *Annals of Pure and Applied Logic*, 163(8):1108 – 1139, 2012. Continuity, Computability, Constructivity: From Logic to Algorithms.

Appendix A Computability on the Baire space

Property 3.2.3: Computability $\iff$ Effective continuity

Let $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$ be a map. Then F is computable if and only if it is effectively continuous.

Proof.

$\implies$: Suppose F computable. There exists a Turing machine M that computes F. And we show that for any $\sigma \in \mathbb{N}^*$, there exists a recursively enumerable set $\mathcal{A}'$ such that $F^{-1}([\sigma]) = \bigcup_{\tau' \in \mathcal{A}'}[\tau']$.

To do so, let $\sigma \in \mathbb{N}^*$. Suppose we have an effective enumeration of $\mathbb{N}^*$, we try to compute $\{F(\tau \cdot \#^\omega)(n)\}_{n \le |\sigma|}$ (where $\#$ is a symbol added to the oracle of M such that if M reads the value $\#$ on the oracle, it stops). Define $\mathcal{A}' = \{\tau' \in \mathbb{N}^* : \sigma \sqsubseteq M^{\tau' \cdot \#^\omega}(0) \cdot \ldots \cdot M^{\tau' \cdot \#^\omega}(|\sigma| - 1)\}$. $\mathcal{A}'$ is recursively enumerable (we compute in parallel on all the τ' of $\mathbb{N}^*$ that have already been enumerated). We have, of course, that $\bigcup_{\tau' \in \mathcal{A}'} \tau' \subseteq F^{-1}([\sigma])$.

Let $p \in F^{-1}([\sigma])$. As $F(p) \in [\sigma]$, the computation of the $|\sigma|$ first values $F(p)(0), ..., F(p)(|\sigma| - 1)$ by M is done in finite time, M can only call a finite number of time the oracle p during its computations. Let n be the greater index read on the oracle. Then $F([p|n + 1]) \subseteq [\sigma]$, and so $(p|n + 1) \in \mathcal{A}'$. Finally, $p \in \bigcup_{\tau' \in \mathcal{A}'}[\tau']$.

We have obtained that $F^{-1}([\sigma]) = \bigcup_{\tau' \in \mathcal{A}'}[\tau']$. We conclude that F is effectively continuous because a single machine could recursively enumerate the $\mathcal{A}'_\sigma$ by enumerating all the $\sigma \in \mathbb{N}^*$.

$\impliedby$: Suppose F is effectively continuous, and let $p \in \mathcal{N}$ and $i \in \mathbb{N}$. We show there exists a Turing machine that computes $F(p)(i)$ uniformly in p and i.

Define M that recursively enumerate such a set $\mathcal{A}$ (cf. definition 3.2.2) until it finds a couple $(\sigma, \tau) \in \mathcal{A}$ such that $[\tau] \sqsubseteq p$ (you can verify that by calling on the oracle p) and $|\sigma| \ge i + 1$. If M finds such a couple (σ, τ), because one has $F(p)(i) = \sigma(i)$, M only has to return $\sigma(i)$. □

Property 3.2.4: Computability with oracle $\iff$ Continuity

Let $F : \mathcal{N} \mapsto \mathcal{N}$. Then F is continuous if and only if there exists an oracle $\mathcal{A} \subseteq \mathbb{N}$ such that F is computable relatively to $\mathcal{A}$. In other words, if and only if there exists a Turing machine M and a set $\mathcal{A} \subseteq \mathbb{N}$ such that: $\forall p \in \text{dom}(F), \forall i \in \mathbb{N} \quad F(p)(i) = M^{\mathcal{A},p}(i)$

Proof.

$\impliedby$: Suppose F to be computable relatively to an oracle $\mathcal{A}$, and let $p \in \mathcal{N}$. Then for any m, the computation of the finite sequence $\{F(p)(i)\}_{i \le m}$ calls on finitely many values on the oracle p: let n be the greatest index of those called values. Then for any p' such that $(p|n) = (p'|n)$, one has $F(p)(i) = F(p')(i)$ for $i \le m$.

We conclude that, $[p|n] \subseteq F^{-1}(F(p)|m)$, which means that F is continuous.

$\implies$: Conversely suppose F is continuous, and define

$$\mathcal{A} = \left\{(\tau, \sigma) \in \mathbb{N}^* : [\tau] \subseteq F^{-1}([\sigma])\right\}.$$

And by a similar procedure than in the previous proof (a "big enough" subset of $\mathcal{A}$ was recursively enumerable before; now we use it as an oracle), F is computable by a Turing machine M equipped with $\mathcal{A}$ as an oracle.

An attentive reader may have noticed we claimed that $\mathcal{A} \subseteq \mathbb{N}$. This is indeed true, because there is an effective bijection between $\mathbb{N}$ and $\mathbb{N}^*$: we could have manipulated in the previous proof an encoding of $\mathcal{A}$ in $\mathbb{N}$, rather than $\mathcal{A}$ itself. □

Appendix B Effective cb$_0$ topological spaces

Property 4.1.6: Properties of the standard representation

The standard representation δ_X of an effective cb$_0$ space $(X, \mathcal{B})$ is:

1. admissible
2. effectively continuous
3. effectively open
4. and it computably realizes effectively continuous maps.

Proof.

1. δ_X is surjective. Indeed, as X is a T_0 space, each point of X is characterized by the set of open neighborhood that contain it; in other words, e is an homeomorphism between X and its image.
2. δ_X is effectively continuous. Let B_i be an open of $\mathcal{B}$, then:
$$\delta_X^{-1}(B_i) = \bigcup_{n \in \mathbb{N}} \{p \in \mathbb{N} : p(n) = i+1\}$$
So it is an effective open set.
3. δ_X is effectively open. Indeed, let $[\sigma]$ be a cylinder of $\mathcal{N}$. Then:
$$\delta_X([\sigma]) = \bigcap_{i+1 \in \sigma} B_i$$
Which is an effective open set, as it is a finite intersection of effective open sets.
4. δ_X is admissible and computably realize effectively continuous maps. In this proof, we use the notations of definition 4.1.5.
   (a) Effective realization: let $f :\subseteq \mathcal{N} \mapsto X$ be an effectively continuous map. We show there exists a computable map $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$ such that $f = \delta_X \circ F$.
   As f is effectively continuous, the sets $\{f^{-1}(B_i)\}$ are uniform effective open sets. In other words, the following set is recursively enumerable by a machine M':
   $$\mathcal{A} = \left\{(i, \sigma) \in \mathbb{N} \times \mathbb{N}^* : [\sigma] \subseteq f^{-1}(B_i)\right\}$$
   We additionally notice that:
   $$\forall i \in \mathbb{N}, \forall p \in \mathrm{dom}(f), f(p) \in B_i \iff \exists k, f(p|k) \subseteq B_i$$
   We now define a computable map F associated to the machine M which realizes in the Baire space the map f. To do so, let $p \in \mathrm{dom}(f)$ and $n \in \mathbb{N}$. We define the value $M^p(n)$ as being $\pi_1(u)$, where u is the n^{th} value of $\mathcal{A}$ returned by M' such that $\pi_2(u) \sqsubseteq p$.
   Then the map $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$ associated to M is computable and such that:
   $$\{i \in \mathbb{N} : \exists n, M^p(n) = F(p)(n) = i+1\} = \{i \in \mathbb{N} : \exists \sigma \sqsubseteq p, f([\sigma]) \subseteq B_i\}$$
   To rephrase this, $e \circ f(p) = \rho \circ F(p)$, which leads to: $f = \delta_X \circ F$.
   (b) Admissibility: Suppose $f :\subseteq \mathcal{N} \mapsto X$ is a continuous map. Then in a similar way to the effective realization, we show there exists a Turing machine M such that, if $\mathcal{A}$ (the previous set) is given as an oracle, then for $p \in \mathrm{dom}(f)$ one has:
   $$\left\{i \in \mathbb{N} : \exists n, M^{\mathcal{A},p}(n) = i+1\right\} = \{i \in \mathbb{N} : f(p) \in B_i\}$$
   To rephrase this, the map F associated to $M^{\mathcal{A}}$ is continuous (according to property 3.2.4) and such that $f = \delta_X \circ F$.

□

Property 4.2.1: Admissible representations and final topology

Let $(X, \mathcal{B})$ be a cb$_0$ topological space. For any admissible representation δ of X, the final topology with respect to δ is $\mathcal{O}(X)$.

Proof.

Let $O \subseteq X$ be a subset of X.

$\implies$: Suppose $O \in \mathcal{O}(X)$. Then because δ is continuous, $\delta^{-1}(O)$ is an open set.

$\impliedby$: Suppose that $\delta^{-1}(O)$ is an open set. As δ is admissible, there exists a continuous map g such that $\delta_X = \delta \circ g$. So $\delta_X^{-1}(O)$ is also an open set. And because δ_X is an open map, O is also an open set. □

Property 4.2.2: (δ_X, δ_Y)-continuity and topological continuity

Let $(X, \mathcal{B}^{(X)})$ and $(Y, \mathcal{B}^{(Y)})$ be two cb$_0$ topological spaces. For any map $f :\subseteq X \mapsto Y$, the two following conditions are equivalent:

1. f is continuous.
2. f is (δ_X, δ_Y)-continuous.

Proof.

Let $f :\subseteq X \mapsto Y$ be a map.

$\implies$: Suppose f is continuous. Then $f \circ \delta_X$ is also a continuous map, from $\mathcal{N}$ to Y. As δ_Y is admissible, there exists a continuous map $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$ such that:

$$\forall p \in \text{dom}(f \circ \delta_X), f \circ \delta_X(p) = \delta_Y \circ F(p)$$

Which means that f is (δ_X, δ_Y)-continuous.

$\impliedby$: Suppose $f :\subseteq X \mapsto Y$ is (δ_X, δ_Y)-continuous. There exists a continuous map $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$ such that $f \circ \delta_X = \delta_Y \circ F$. Let O be an open set of Y.
Then $F^{-1} \circ \delta^{(Y)-1}(O)$ is an open set by continuity of F and δ_Y, so $\delta_X^{-1}(f^{-1}(O))$ is also open by the equality just above. As δ_X is admissible, its final topology is the topology induced by $\mathcal{B}^{(X)}$. We conclude that $f^{-1}(O)$ is also an open set, which means that f is continuous. □

Property 4.2.3: (δ_X, δ_Y)-computability and effective continuity

Let $(X, \mathcal{B}^{(X)})$ and $(Y, \mathcal{B}^{(Y)})$ be two effective cb$_0$ topological spaces. For any map $f :\subseteq X \mapsto Y$, the two following conditions are equivalent:

1. f is (δ_X, δ_Y)-computable.
2. f is effectively continuous.

Proof.

The proof is very similar to the one of property 4.2.2. □

Property 4.3.3: Inclusions in the Borel hierarchy

1. For any $1 \leq \beta < \omega_1$,
$$\boldsymbol{\Sigma}^0_{\boldsymbol{\beta}} \cup \boldsymbol{\Pi}^0_{\boldsymbol{\beta}} \subseteq \boldsymbol{\Delta}^0_{\boldsymbol{\beta+1}}$$
2. For any $b \in O$,
$$\Sigma^0_{(b)} \cup \Pi^0_{(b)} \subseteq \Delta^0_{(b+1)}$$

Proof.

1. Let β such that $1 \leq \beta < \omega_1$. We show the announced inclusion.
For any $A \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}$ one has $A = A \setminus \emptyset$, and so $A \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta+1}}$. Additionally, for any $A \in \boldsymbol{\Pi}^0_{\boldsymbol{\beta}}$ one has $A = X \setminus A^c$, and by the first case $X \in \boldsymbol{\Sigma}^0_{\boldsymbol{1}} \subseteq \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}$ and $A \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta+1}}$.
Similarly, for any $A \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}$ one has $A^c \in \boldsymbol{\Pi}^0_{\boldsymbol{\beta}}$, and by the previous case $A^c \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta+1}}$. We conclude that $A \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta+1}}$. Similarly, for any $A \in \boldsymbol{\Pi}^0_{\boldsymbol{\beta}}$ one has $A^c \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}} \subseteq \boldsymbol{\Sigma}^0_{\boldsymbol{\beta+1}}$, and so $A \in \boldsymbol{\Pi}^0_{\boldsymbol{\beta+1}}$. Which concludes the proof.
2. The demonstration is very similar to the one above, one just has to mind uniformity.

□

Property 4.3.4: Stability in the Borel hierarchy

1. For any $1 \leq \beta < \omega_1$, $\boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}$ is stable under countable unions and finite intersections.
2. For any $b \in O$, $\Sigma^0_{(b)}$ is stable under uniform union and finite intersections.

Proof.

1. Let $1 \leq \beta < \omega_1$. $\boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}$ is obviously stable under countable union. For finite intersections, we show this property by induction.
   (a) The case of $\boldsymbol{\Sigma}^0_{\boldsymbol{1}}$ is part of the definition of open sets.
   (b) Suppose this property holds for any $\beta' < \beta$, and let $A, B \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}$. We write:
   $$A = \bigcup_{i \in \mathbb{N}} A_i \setminus A'_i \quad \text{where } A_i, A'_i \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta_i}}, \beta_i < \beta$$
   and similarly $B = \bigcup_{i \in \mathbb{N}} B_i \setminus B'_i$. Then:
   $$A \cap B = \bigcup_{i,j \in \mathbb{N}} (A_i \cap B_i) \setminus (A'_i \cup B'_i)^c$$
   By applying the induction hypothesis and property 4.3.3, we conclude $A \cap B \in \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}}$.
2. The demonstration is very similar to the one above, one just has to mind uniformity.

□

Property 4.3.7: Inclusions in the difference hierarchy

Boldface hierarchy:

1. For any $1 \leq \alpha < \omega_1$ and $1 \leq \beta < \omega_1$,

$$D_\alpha(\mathbf{\Sigma^0_\beta}) \subseteq D_{\alpha+1}(\mathbf{\Sigma^0_\beta})$$

2. For any $1 \leq \beta < \omega_1$,

$$\bigcup_{\alpha<\omega_1} D_\alpha\left(\mathbf{\Sigma^0_\beta}\right) \subseteq \mathbf{\Delta^0_{\beta+1}}$$

Lightface hierarchy:

1. For any $a, b \in O$,

$$D_{(a)}(\Sigma^0_{(b)}) \subseteq D_{(a+1)}(\Sigma^0_{(b)})$$

2. For any $b \in O$,

$$\bigcup_{a\in O} D_{(a)}\left(\Sigma^0_{(b)}\right) \subseteq \Delta^0_{(b+1)}$$

Proof.

We do the proof of the Boldface case, as the Lightface case is again very similar:

1. The inclusion $D_\alpha \subseteq D_{\alpha+1}$ is obvious.
2. We demonstrate this property by showing that:
   (a) For any $1 \leq \alpha < \omega_1$ and family of increasing sets $\mathbf{\Sigma^0_\beta}$ $\{A_\gamma\}_{\gamma<\alpha}$, one has:

$$D_\alpha(\{A_\gamma\}_{\gamma<\alpha}) = \bigcup_{\gamma<\alpha, r(\gamma)\neq r(\alpha)} A_\gamma \setminus \bigcup_{\theta<\gamma} A_\theta$$

   Which means $D_\alpha(\{A_\gamma\}_{\gamma<\alpha}) \in \mathbf{\Sigma^0_{\beta+1}}$.

   (b) Furthermore, with the same notations, one has:

$$X \setminus D_\alpha(\{A_\gamma\}_{\gamma<\alpha}) = D_{\alpha+1}(\{A'_\gamma\}_{\gamma<\alpha+1})$$

   where $A'_\gamma = A_\gamma$ if $\gamma \neq \alpha$, and $A'_\alpha = X$. Which means $\tilde{D}_\alpha(\mathbf{\Sigma^0_\beta}) \in D_{\alpha+1}(\mathbf{\Sigma^0_\beta}) \subseteq \mathbf{\Sigma^0_{\beta+1}}$ (by (a)).

   (c) So we conclude that for any $1 \leq \alpha, \beta < \omega_1$, one has:

$$D_\alpha\left(\mathbf{\Sigma^0_\beta}\right) \subseteq \mathbf{\Delta^0_{\beta+1}}$$

□

Appendix C Proofs of the results

C.1 Results: hierarchies on cb_0 spaces

In this section, we demonstrate the following theorem:

[Original result] - Theorem 5.2.1: Effective equivalence of topological and algorithmic complexities

Let $(X, \mathcal{B})$ be an effective cb_0 space. For any $a, b \in O$ and $S \subseteq X$:

$$S \in D_{(a)}\left(\Sigma^0_{(b)}(X)\right) \iff \delta_X^{-1}(S) \in D_{(a)}\left(\Sigma^0_{(b)}(\text{dom}(\delta_X))\right)$$

We divide the proof in three parts. The first will be the direct implication, which only consists in an induction. We develop several useful lemma in the second part, before proving the converse implication in the third and last part.

C.1.1 Proof of the direct implication

This proof consists in an induction on $b \in O$, and relies on the effective continuity of δ_X. We then conclude by directly reasoning over a.

Case $|a|_O = |b|_O = 1$:
Let $S \in \Sigma^0_1(X)$: then S is an effective open set of X. As δ_X is effectively continuous, $\delta_X^{-1}(S)$ is an effective open set of $\text{dom}(\delta)$. Which is exactly the result we wanted, $\delta_X^{-1}(S) \in \Sigma^0_1(\text{dom}(\delta))$.

Case $|a|_O = 1, |b|_O > 1$:
Let $a, b \in O$ such that $|a|_O = 1$ and $|b|_O > 1$. Suppose by induction that the property is true for any $b' <_o b$: we demonstrate the property is still verified for a and b. To do so, let $S \in \Sigma^0_{(b)}(X)$. We have:

$$S = \bigcup_{n \in \mathbb{N}} S_n \setminus S'_n, \qquad \text{where } S_n, S'_n \in \Sigma^0_{(b_n)}(X) \text{ uniformly and } b_n <_o b$$

Then:

$$\delta^{-1}(S) = \bigcup_{n \in \mathbb{N}} \delta^{-1}(S_n) \setminus \delta^{-1}(S'_n), \qquad \text{where } S_n, S'_n \in \Sigma^0_{(b_n)}(X) \text{ uniformly and } b_n <_o b$$

There, we can conclude by induction hypothesis on each $b_n <_o b$.

Proof for any $a \in O$:
Let $a \in O$ and $b \in O$. Let $S \in D_{(a)}\left(\Sigma^0_{(b)}(X)\right)$. We have:

$$S = D_{|a|_O}\left(\{S_{(c)}\}_{c <_o a}\right)$$

The previous result for $a = 1$ yields:

$$\delta^{-1}(S) = D_{|a|_O}\left(\{\delta^{-1}(S_{(c)})\}_{c <_o a}\right) \in D_{(a)}\left(\Sigma^0_{(b)}(\text{dom}(\delta))\right)$$

C.1.2 Preliminary lemmas

We demonstrate several preliminary lemmas in this section. The first of them asserts that a non-meager Borel set necessarily is co-meager somewhere.

Definition C.1.1: Baire property

For a topological space X, a subset $A \subseteq X$ has the **Baire property** if there exists an open set U such that $A \Delta U$ is meager in X.

Property C.1.2: Baire property on Borel sets

Any Borel set of a topological space X has the Baire property.

Proof. The Baire property is stable under countable union and intersection. Furthermore, it is stable under complement: indeed, if $A \subseteq X$ and $U \in \mathcal{O}(X)$ are two subsets such that $A \Delta U$ is meager, then as $\overline{U} \setminus U$ is meager, we obtain that $A \Delta \overline{U} \subseteq (A \Delta U) \cup (A^c \cap (\overline{U} \setminus U))$ is meager, and so that $A^c \Delta \overline{U}^c = A \Delta \overline{U}$ is meager too. We conclude that the subsets of X verifying the Baire property form a σ-algebra that contains the open sets of X, which yields that any Borel sets has the Baire property. □

Lemma C.1.3: The Baire property on a Baire space

Let X be a Baire space (ie. a space on which Baire category theorem is verified). Suppose that $A \subseteq X$ has the Baire property: then two exclusive cases are possible.

1. A is meager.
2. Or there exists an open set $U \neq \emptyset$ such that A is co-meager in U.

Proof. Suppose that $M = A \Delta U$ is meager. If A is non-meager, then U is non-empty. And because $U \setminus A \subseteq M$, A is co-meager in U.

Furthermore, the two possibilities cannot occur simultaneously. Indeed, if A were meager and $U \neq \emptyset$ were an open set such that $U \setminus A$ were meager, then $U \subseteq A \cup (U \setminus A)$ would be meager and non-empty, which is absurd in a Baire space. □

Now, we focus on the fibers by the standard representation:

Lemma C.1.4: $\delta_X^{-1}(x)$ is a Polish space

Let $(X, \mathcal{B})$ be an effective cb_0 space equipped with the standard representation δ_X. Then for any $x \in X$, $\delta_X^{-1}(x)$ is a $\mathbf{\Pi}_2^0$ subset of $\mathcal{N}$. As such, it is a Polish space, and in particular a Baire space.

Proof. Let $\mathcal{B} = (B_i)_{i \in \mathbb{N}}$ be the countable basis of X we based the standard representation on, and let $x \in X$ be an element of X. We prove that $\delta_X^{-1}(x)$ is a $\mathbf{\Pi}_2^0$ subset of $\mathcal{N}$. By [Kec95], Theorem 3.11, it entails that $\delta_X^{-1}(x)$ is a Polish space.
In the way we defined δ_X, define $e(x) = \{i \in \mathbb{N} : x \in B_i\} \subseteq \mathbb{N}$. We have:

$$\delta_X^{-1}(x) = \left(\bigcap_{i \in e(x)} \{p \in \mathcal{N} : \exists n, p(n) = i + 1\} \right) \cap \{p \in \mathcal{N} : \forall n, p(n) + 1 \in e(x)\}$$

This is a finite or countable intersection of open sets of $\mathcal{N}$ with a closed set, and as such a an element of $\mathbf{\Pi}_2^0(\mathcal{N})$. □

The following lemma is necessary to rewrite the hierarchies on $\mathrm{dom}(\delta_X)$ in their standard way:

Lemma C.1.5: Rewriting the Lightface hierarchy on $\mathcal{N}$

Let $T \subseteq \mathcal{N}$ be a subset of $\mathcal{N}$, and let $b \in O$ such that $|b|_O > 1$. Then:

$$\Sigma^0_{(b)}(T) = \left\{ \bigcup_{n\in\mathbb{N}} A_n : A_n \in \Pi^0_{b_n}(T) \text{ uniformly and } b_n <_o b \right\}$$

Proof. The proof is an induction on $b \in O$.

1st case: $|b|_O = 2$: Let $B \in \Sigma^0_2(T)$:

$$B = \bigcup_{n\in\mathbb{N}} A_n \setminus A'_n \quad \text{where } A_n, A'_n \in \mathcal{O}_{\text{eff}} \text{ uniformly}$$

First, for any $n \in \mathbb{N}$, there exists a recursively enumerable subset $W_n \subseteq \mathbb{N}$ such that $A_n = \bigcup_{e\in W_n}[\sigma_e]$, where $\{\sigma_e\}_{e\in\mathbb{N}}$ is an enumeration of $\mathbb{N}^*$. Then one has:

$$A_n \setminus A'_n = \left(\bigcup_{e\in W_n} [\sigma_e] \cap A'^{\,\mathrm{c}}_n \right) \cap T$$

Which is a uniform union of elements in $\Pi^0_1(T)$ (as $[\sigma_e]$ are effective clopen sets). So, B is written as the countable union of elements of complexity Π^0_1, which is the desired result.

Case $|b|_O > 2$: Suppose by induction that the property is true for any $b' <_o b$. We demonstrate that the property is true at rank b. To do so, let $B \in \Sigma^0_{(b)}(T)$:

$$B = \bigcup_{n\in\mathbb{N}} A_n \setminus A'_n \quad \text{where } A_n, A'_n \in \Sigma^0_{(b_n)}(T) \text{ uniformly and } b_n <_o b$$

Let $n \in \mathbb{N}$. If $|b_n|_O = 1$, we apply the method of the first case. Otherwise, applying the induction hypothesis on A_n yields:

$$A_n = \bigcup_{i\in\mathbb{N}} C_{n,i} \quad \text{where } C_{n,i} \in \Pi^0_{(b_{n,i})}(T) \text{ uniformly and } b_{n,i} <_o b_n$$

Which leads to:

$$A_n \setminus A'_n = \bigcup_{i\in\mathbb{N}} C_{n,i} \cap A'^{\,\mathrm{c}}_n$$

And as $C_{n,i} \in \Pi^0_{(b_{n,i})} \subseteq \Pi^0_{b_n}$ and $A'^{\,\mathrm{c}}_n \in \Pi^0_{(b_n)}$, we conclude that B is a countable union of a uniform family of effective $\Pi^0_{(b_n)}$ sets, with $b_n <_o b$. Which is exactly the desired result. ☐

The last lemma is the cornerstone of the converse implication, and is largely inspired by the transformation in the proof of Lemma 17 in [Ray07]. Effectivizing these results and transferring them on the Lightface version of the Borel hierarchy were two mains contributions of this internship:

[Original result] - Lemma 5.3.1: Modified Lemma 17 of [Ray07]

Let $(X, \mathcal{B})$ be an effective cb_0 space. For any $b \in O$ and $A \subseteq \mathrm{dom}(\delta_X)$,

$$A \in \Sigma^0_{(b)}(\mathrm{dom}(\delta_X)) \implies B(A) \in \Sigma^0_{(b)}(X)$$

Additionally, the transformation preserves uniformity for families of subsets in $\mathrm{dom}(\delta_X)$.

Proof.

We do this proof by induction on $b \in O$.

1$^{\text{st}}$ case: $|b|_O = 1$:

Let $A \in \Sigma^0_1(\text{dom}(\delta_X))$. We show that $B(A) \in \Sigma^0_1(X)$.
To do so, let $x \in B(A)$. As A is an effective open set of $\text{dom}(\delta_X)$, $\delta_X^{-1}(x) \cap A$ is in particular an open set of $\delta^{-1}(x)$ (which is a Baire space by lemma C.1.4). So $\delta_X^{-1}(x) \cap A$ is non-meager if and only if it is non-empty, ie. $x \in B(A)$ if and only if $\delta_X^{-1}(x) \cap A$ is not empty.
From there, we conclude that $B(A) = \delta_X(A)$. And because δ_X is an effectively open representation, we obtain $B(A) = \delta_X(A) \in \Sigma^0_1(\text{dom}(\delta_X))$.

Inductive case:

Suppose that for any $b' <_o b$, and for any $\{A^{(i)}\}_{i \in O}$ uniform family of effective sets in $\Sigma^0_{(b')}(\text{dom}(\delta_X))$, $\{B(A^{(i)})\}_{i \in O}$ is a uniform family of elements in $\Sigma^0_{(b')}(X)$. We now prove this property holds also at rank (b). To do so, let $A \in \Sigma^0_b(\text{dom}(\delta_X))$. By lemma C.1.5, the Borel hierarchy is the same as its "traditional" form and we write:

$$A = \bigcup_{n \in \mathbb{N}} A_n, \qquad \text{where } A_n \in \Pi^0_{(b_n)} \text{ and } b_n <_o b$$

We have $x \in B(A)$ if and only if there exists $n \in \mathbb{N}$ such that $\delta_X^{-1}(x) \cap A_n$ is non-meager in $\delta_X^{-1}(x)$.
As A_n is an effective Borel set (and so, a Borel set) of $\text{dom}(\delta_X)$ (and so of $\delta_X^{-1}(x)$ for the induced topology), and that $\delta_X^{-1}(x)$ is a Baire space by lemma C.1.4, by applying lemma C.1.3 and property C.1.2, $x \in B(A)$ if and only if there exists $n \in \mathbb{N}$ and a non empty cylinder $[\sigma]$ such that $A_n \cap \delta_X^{-1}(x)$ is co-meager in $[\sigma] \cap \delta_X^{-1}(x)$.
In other words: if and only if $(A_n \cup [\sigma]^{\text{c}}) \cap \delta_X^{-1}(x)$ is co-meager in $\delta_X^{-1}(x)$, and $[\sigma] \cap \delta_X^{-1}(x) \neq \emptyset$.
Which we rewrite in: if and only if $({A_n}^{\text{c}} \cap [\sigma]) \cap \delta_X^{-1}(x)$ is not non-meager in $\delta_X^{-1}(x)$, and $[\sigma] \cap \delta_X^{-1}(x) \neq \emptyset$.
From there, we conclude that:

$$B(A) = \bigcup_{n \in \mathbb{N}, \sigma \in \mathbb{N}^*} \delta_X([\sigma]) \setminus B({A_n}^{\text{c}} \cap [\sigma])$$

Furthermore, $A_n \in \Pi^0_{(b_n)}(\text{dom}(\delta_X))$, so ${A_n}^{\text{c}} \cap [\sigma] \in \Sigma^0_{(b_n)}(\text{dom}(\delta_X))$. By induction hypothesis, the sets $B({A_n}^{\text{c}} \cap [\sigma])$ form a uniform family of sets in $\Sigma^0_{(b_n)}(X)$, because the sets A_n themselves form a uniform family.
And on the other hand, as δ_X is effectively open, $\delta_X([\sigma])$ is also an effective open set, and as such an element of $\Sigma^0_{(b_n)}$ for any $n \in \mathbb{N}$ by property 4.3.3: this means the sets $\delta_X([\sigma])$ form a uniform family of elements in $\Sigma^0_{(b_n)}$ for any $n \in \mathbb{N}$.

We conclude that $B(A) \in \Sigma^0_{(b)}(X)$. Furthermore, for a uniform family $\{A^{(i)}\}_{i \in O}$ of sets in $\Sigma^0_{(b)}(\text{dom}(\delta_X))$, the writing above assures that the family $\{B(A_i)\}_{i \in O}$ is a uniform family of sets in $\Sigma^0_{(b)}(X)$. Which is what we wanted. ☐

C.1.3 Proof of the converse implication

Let us briefly recall the theorem we want to demonstrate here:

[Original result] - **Theorem 5.2.1: Effective equivalence of topological and algorithmic complexities**

Let $(X, \mathcal{B})$ be an effective cb$_0$ space. For any $a, b \in O$ and $S \subseteq X$:

$$S \in D_{(a)}\left(\Sigma^0_{(b)}(X)\right) \iff \delta_X^{-1}(S) \in D_{(a)}\left(\Sigma^0_{(b)}(\text{dom}(\delta_X))\right)$$

The proof we give is analogous to the one of [dB13], and differs mainly in the use of computable ordinals:

Proof. Suppose there exists $a, b \in O$ such that $\delta_X^{-1}(S) \in D_{(a)}\left(\Sigma^0_{(b)}(\mathrm{dom}(\delta))\right)$. We can write $\delta_X^{-1}(S)$ as:

$$\delta_X^{-1}(S) = D_{(a)}\left(\{A_c\}_{c<_o a}\right)$$

We will conclude that $S \in D_{(a)}\left(\Sigma^0_{(b)}(X)\right)$ by showing (with the transformation of lemma 5.3.1) the following equality:

$$S = D_{(a)}\left(\{B(A_c)\}_{c<_o a}\right)$$

$\subseteq$: Let $x \in S$. Because

$$\delta_X^{-1}(x) = \bigcup_{c<_o a} A_c \cap \delta_X^{-1}(x)$$

there exists $c <_o a$ such that $x \in B(A_c)$. Suppose the opposite: $\delta_X^{-1}(x)$ would be a countable union of meager sets, and would itself be meager by Baire category theorem: but this is absurd, as $\delta_X^{-1}(x)$ is a Baire space. Consider now the smallest (for the strict order $<_o$) $c_x <_o a$ such that $x \in B(A_{c_x})$. We now prove that $r(c_x) \neq r(a)$.
By choice of c_x, for any $d <_o c_x$ $A_d \cap \delta_X^{-1}(x)$ is meager in $\delta_X^{-1}(x)$; this implies $\bigcup_{d<_o c_x}(A_d \cap \delta_X^{-1}(x))$ is meager in $\delta_X^{-1}(x)$. Since $A_{c_x} \cap \delta_X^{-1}(x)$ is non-meager in $\delta_X^{-1}(x)$, we deduce that

$$\left(A_{c_x} \setminus \bigcup_{d<_o c} A_d\right) \cap \delta_X^{-1}(x)$$

is non-meager in $\delta_X^{-1}(x)$. In particular, this set is non-empty and as such contains an element p. By hypothesis

$$p \in \delta_X^{-1}(x) \subseteq \delta_X^{-1}(S) = D_{(a)}\left(\{A_c\}_{c<_o a}\right)$$

we have $r(c_x) \neq r(a)$. Which is exactly what we wanted to show.

$\supseteq$: Conversely, let $y \in D_{(a)}\left(\{A_c\}_{c<_o a}\right)$. There exists $c_y <_o a$ such that $r(c_y) \neq r(a)$ and

$$y \in B(A_{c_y}) \setminus \bigcup_{d<_o c_y} B(A_d)$$

Then $A_{c_y} \cap \delta_X^{-1}(y)$ is non-meager in $\delta_X^{-1}(y)$, and $\bigcup_{d<c_y} A_d \cap \delta_X^{-1}(y)$ is meager in $\delta_X^{-1}(y)$. This yields

$$\left(A_{c_y} \setminus \bigcup_{d<_o c_y} A_d\right) \cap \delta_X^{-1}(y)$$

is non-meager in $\delta_X^{-1}(y)$, and in particular is non-empty. Since $r(c_y) \neq r(a)$, this implies there exists $p \in \delta_X^{-1}(y)$ such that $p \in D_a(\{A_c\}_{c<_o a})$. So $p \in \delta_X^{-1}(S)$, and $y \in S$. $\square$

C.2 Results: piecewise-computability

C.2.1 Preliminary properties

In this section, we introduce several computational properties on the Baire space we will use later on.

Property C.2.1: Dense sets

Let $A \subseteq \mathcal{N}$ be a dense subset of $\mathcal{N}$, and $F : A \mapsto \mathcal{N}$ be a computable constant map. Then the constant $F(A)$ is computable.

Proof.

Three cases can happen when a Turing machine computes on the input $p \in \mathcal{N}$:

1. $p \in \text{dom}(F)$: the Turing machine computes correctly.
2. $p \notin \text{dom}(F)$, and there exists $n \in \mathbb{N}$ such that the machine loops while computing $F(p)(n)$.
3. $p \notin \text{dom}(F)$, and there exists $n \in \mathbb{N}$ such that the machine "errs" and $F(p)(n)$ is different from $F(A)(n)$.

Because A is dense, the two last cases are not possible. Indeed, if $O = \mathbb{N}^n \cdot n_1 \cdot ... \cdot n_p \cdot \mathcal{N}$ (for some p), then O is an open set. So $A \cap O$ is non-empty, and the machine cannot errs while reading the values $n_1, ..., n_p$, neither loop.

So, by testing in parallel an enumeration of the cylinders of $\mathcal{N}$, it is possible to determine the j first values $\{M^p(0), ..., M^p(j)\}$ for $j \in$ (by choosing p big enough): and those values can only be those of $F(A)(0), ..., F(A)(j)$. ☐

Property C.2.2: Arbitrary sets

Let $A \subseteq \mathcal{N}$ be a subset such that $\{\sigma \in \mathbb{N}^* : [\sigma] \cap A \neq \emptyset\}$ is recursively enumerable, and $F :\mapsto \mathcal{N}$ be a computable constant map. Then the constant $F(A)$ is computable.

Proof. The proof is very similar to the previous one (property C.2.1): rather than testing in parallel an enumeration of the cylinders of $\mathcal{N}$, one should test the enumeration of $\{\sigma \in \mathbb{N}^* : [\sigma] \cap A \neq \emptyset\}$. ☐

Lastly, here is another property that we will use frequently:

Property C.2.3: Sequential convergence and admissible representations

Let $(X, \mathcal{B})$ be an effective cb_0 space equipped with an admissible representation δ, and $(x_n)_{n\in\mathcal{N}}$ be a sequence of X. Then the following assertions are equivalent:

1. The sequence $(x_n)_{n\in\mathbb{N}}$ converges towards $x \in X$.
2. The map f defined by $f(0^n 10^\omega) = x_n$ and $f(0^\omega) = x$ is continuous.
3. There exists a sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathcal{N}$ such that for any $n \in \mathbb{N}$, $\delta(p_n) = x_n$ and the sequence $(p_n)_{n\in\mathbb{N}}$ converge towards an element $p \in \mathcal{N}$ such that $\delta(p) = x$.

Proof.

$1 \iff 2$: The convergence of a sequence can always be written as the continuity of such a map.

$2 \iff 3$: As the representation δ is admissible, by property 4.2.2 the map f is continuous if and only if there exists a continuous map $F : \{0^n 10^\omega : n \in \mathbb{N}\} \cup \{0^\omega\} \mapsto \mathcal{N}$ such that $\delta \circ F(0^n 10^\omega) = f(0^n 10^\omega)$ and $\delta \circ F(0^\omega) = f(0^\omega)$. To rephrase this, by equivalence $1 \iff 2$, iff there exists a sequence of names $(p_n)_{n\in\mathbb{N}}$ converging towards p such that $\delta(p_n) = x_n$ and $\delta(p) = x$. ☐

C.2.2 Presentation of the results and their proofs

We give here a proof for one of the original results we obtained during this internship. As we said earlier, the proof of the non-trivial implication uses the transformations $A \mapsto B(A)$ we introduced in the previous section.

[Original result] - Theorem 6.2.1: Piecewise-computability: countable cover

Let $(X, \mathcal{B}^{(X)})$ and $(Y, \mathcal{B}^{(Y)})$ be two cb_0 topological spaces and $f :\subseteq X \mapsto Y$ be a map. The two following assertions are equivalent:

1. There exists a cover $\{P_i\}_{i\in\mathbb{N}}$ of $\text{dom}(\delta_X)$ of complexity $\Sigma^0_{(b)}(\mathcal{N})$ and a family of computable maps $\{F_i\}_{i\in\mathbb{N}}$ such that:
$$\forall i \in \mathbb{N}, \forall p \in \text{dom}(F) \cap P_i, \qquad \delta_Y \circ F_i(p) = f \circ \delta_X(p)$$
2. There exists a cover $(Q_i)_{i\in\mathbb{N}}$ of X of complexity $\Sigma^0_{(b)}(X)$ such that for any $i \in \mathbb{N}$, $f_{|Q_i}$ is a computable map.

Proof.

$2 \implies 1$: Indeed, if $(Q_i)_{i\in\mathbb{N}}$ is a cover of X of complexity $\boldsymbol{\Sigma}^0_{(b)}$ verifying the computational conditions on f, let $P_i = \delta_X^{-1}(B_i)$. By definition of computability, we obtain the assertion 1.

$1 \implies 2$: Two cases arise.

1st case: If $|b|_O = 1$

By defining $Q_i = \delta_X(P_i)$, we obtain a cover of X that verifies the desired property. Indeed, for any $x \in Q_i$, as $\{\sigma \in \mathbb{N}^* : [\sigma] \cap P_i \cap \delta_X^{-1}(x) \neq \emptyset\} = \{\sigma \in \mathbb{N}^* : x \in \delta_X([\sigma] \cap P_i)\}$, and as δ_X is effectively open, we have that those sets are recursively enumerable. By property C.2.2, $f(x)$ is computed as the constant value of $\delta_Y \circ F_i$ on $\delta_X^{-1}(x) \cap P_i$.

2nd case: If $|b|_O > 1$

We write:
$$P_i = \bigcup_{i\in\mathbb{N}} A_j^{(i)}, \qquad \text{where } A_j^{(i)} \in \Pi^0_{(b_i)}(\mathcal{N}) \text{ uniformly and } |b_i|_O < |b|_O$$

And we define, for any $i, j \in \mathbb{N}$ and $\sigma \in \mathbb{N}^*$ (with the transformation B of lemma 5.3.1
$$C_{i,j,\sigma} = \delta_X([\sigma]) \setminus B({A_j^{(i)}}^{\mathsf{c}} \cap [\sigma])$$

Then we have a cover of X, because:
$$\bigcup_{j,\sigma} C_{i,j,\sigma} = B(A_i) \quad \text{and} \quad \bigcup_{i\in\mathbb{N}} B(A_i) = B\left(\bigcup_{i\in\mathbb{N}} A_i\right) = B(\text{dom}(\delta_X)) = X$$

Additionally, for any $x \in C_{i,j,\sigma}$ one has that $\delta_X^{-1}(x) \cap \left(A_j^{(i)}\right)^{\mathsf{c}}$ is not non-meager in $\delta_X^{-1}(x)$. In particular, $\delta_X^{-1}(x) \cap A_j^{(i)}$ is co-meager, and so dense in $\delta_X^{-1}(x)$. Define now $D_{i,j,\sigma} = [\sigma] \cap \delta_X^{-1}(x) \cap A_j^{(i)}$.
We have that $\{\sigma' \in \mathbb{N}^* : [\sigma'] \cap [\sigma] \cap \delta_X^{-1}(x) \neq \emptyset\} = \{\sigma' \in \mathbb{N}^* : [\sigma'] \cap D_{i,j,\sigma} \neq \emptyset\}$. Furthermore, we know that $\{\sigma' \in \mathbb{N}^* : [\sigma'] \cap [\sigma] \cap \delta_X^{-1}(x) \neq \emptyset\} = \{\sigma' \in \mathbb{N}^* : x \in \delta_X([\sigma'] \cap [\sigma])\}$ is a recursively enumerable set, because δ_X is an effective open map. We obtain that $D_{i,j,\sigma}$ is a set such that $\{\sigma' \in \mathbb{N}^* : [\sigma'] \cap D_{i,j,\sigma} \neq \emptyset\}$ is recursively enumerable, and on which $\delta_Y \circ F_i$ is constant.
By property C.2.2, we conclude that $f(x)$ is computable as the constant value of $\delta_Y \circ F_i$ on the set $D_{i,j,\sigma}$.
□

For the case $(b) = 1$, it is possible to conclude by effective openness of the standard representation. It is an already known proof we mention for the sake of exhaustiveness.

Theorem 6.3.1: Piecewise-computability : finite cover of Σ_1^0 and Π_1^0 sets

With the same notations, the two following assertions are equivalent:

1. There exists a cover $\{P_1, P_2 = {P_1}^c\}$ of $\text{dom}(f)$, where $P_1 \in \Sigma_1^0(\mathcal{N})$, and two computable maps $F_i :\subseteq \mathcal{N} \mapsto \mathcal{N}$ such that:
$$\forall i \in \{1,2\}, \forall p \in \text{dom}(F) \cap P_i, \quad \delta_Y \circ F_i(p) = f \circ \delta_X(p)$$
2. There exists a cover $\{Q_1, Q_2\}$ of X, where the complexity of Q_i is the one of P_i, and such that $f_{|Q_1}$ and $f_{|Q_2}$ are computable.

Proof.

$1 \implies 2$: Define $Q_1 = \delta_X(P_1)$ and $Q_2 = {Q_1}^c$: we obtain a cover of X with the desired property. Indeed, for any $x \in Q_1$, as $\{\sigma \in \mathbb{N}^* : [\sigma] \cap P_1 \cap \delta_X^{-1}(x) \neq \emptyset\} = \{\sigma \in \mathbb{N}^* : x \in \delta_X([\sigma] \cap P_1)\}$, and as δ_X is effectively open, we conclude that those sets are recursively enumerable. $f(x)$ is computed as the constant value of $\delta_Y \circ F_i$ on $\delta_X^{-1}(x) \cap P_1$.
Furthermore, for any $x \in Q_2$ one has $\delta_X^{-1}(x) \cap {P_1}^c$ is co-meager in $\delta_X^{-1}(x)$, and so is dense. We also know that $\{\sigma \in \mathbb{N}^* : [\sigma] \cap \delta_X^{-1}(x) \neq \emptyset\}$ is recursively enumerable, and so that $\{\sigma \in \mathbb{N}^* : [\sigma] \cap \delta_X^{-1}(x) \cap P_2 \neq \emptyset\}$ is also r.e. As $\delta_Y \circ F_2$ is constant on $\delta_X^{-1}(x) \cap P_2$, we can compute $f(x)$.

$2 \implies 1$: By defining $P_i = \delta_X^{-1}(Q_i)$, we obtain a cover verifying the assertion 1. □

One of the original results obtained during this internship is the use of the proof of theorem 6.2.1 to demonstrate the following theorem:

[Original result] - Theorem 6.3.2: Piecewise-computability : finite cover of $\Sigma_{(b)}^0$ sets, where $|b|_O \geq 2$

With the same notations,

Suppose there exists a cover $\{P_1, P_2 = {P_1}^c\}$ of $\text{dom}(f)$, where $P_1 \in \Sigma_{(b)}^0(\mathcal{N})$, and two computable maps $F_i :\subseteq \mathcal{N} \mapsto \mathcal{N}$ such that:
$$\forall i \in \{1,2\}, \forall p \in \text{dom}(F) \cap P_i, \quad \delta_Y \circ F_i(p) = f \circ \delta_X(p)$$

Then there exists a countable cover $\{Q_i\}_{i \in \mathbb{N}}$ of X, where $Q_i \in \Sigma_{(b)}^0(X)$, such that for any $i \in \mathbb{N}$, $f_{|Q_i}$ is a computable map.

Proof.

The proof is the case $1 \implies 2$ of theorem 6.2.1. □

It may look unsatisfactory to obtain a countable cover from a finite one, but the following counter-example inspired by [Zie12] show there is no possibility to associate a cover of cardinal two in the topological space from a cover of two elements in the space of name. One should however notice the proof is different from the one already known, as we had to correct a mistake in [Zie12]:

C.2.3 Counter-example in the finite case

Example 6.3.3: Piecewise-computability : counter-example

Define $f : [0,1] \mapsto \mathbb{R}$ the map such that $f(x) = x$ if $x \in \mathbb{Q} \cap [0,1)$
$f(x) = x - 1$ if $x \in ([0,1] \setminus \mathbb{R}) \cup \{1\}$

We map the segment $[0,1]$ to the circle $\mathcal{S}^1$ with the following map: $q : x \mapsto x \mod 1$. There exists a map $\tilde{f} : \mathcal{S}^1 \mapsto \mathbb{R}$ such that $\tilde{f} \circ q = f$. And:

1. There exists computable maps $F_i :\subseteq \mathcal{N} \mapsto \mathcal{N}$ $(i = 1, 2)$ and a set $A \in \Sigma^0_3$ such that, by defining $P_1 = A$ and $P_2 = A^c$:

$$\forall i \in \{1,2\}, \forall p \in P_i \cap \mathrm{dom}(\delta_X), \quad \delta_{\mathbb{R}} \circ F_i(p) = \tilde{f} \circ \delta_{\mathcal{S}^1}(p)$$

2. But there is no partition $\{B_1, B_2\}$ of $\mathcal{S}^1$ that would make f continuous on each of its elements; and *a fortiori* computable.

Proof.

1. We show that $\tilde{f}$ is computably 2-realizable. This is where we correct a mistake in the original proof, as our proof uses an admissible representation (and the original text claimed to do so, but did not).

   To do so, consider the two effective open sets $\delta^{-1}_{\mathcal{S}^1}(]-3/8, 3/8[)$ and $\delta^{-1}_{\mathcal{S}^1}(]1/8, 7/8[)$: their intersection is non-empty, and so it is possible to create a effective clopen (both open and closed) set C of $\mathcal{N}$ such that $\delta^{-1}_{\mathcal{S}^1}(]-1/8, 1/8[) \subseteq C$ and $\delta^{-1}_{\mathcal{S}^1}(]3/8, 5/8[) \subseteq C^c$.

   We now define the set A:

$$A = (C^c \cap \delta^{-1}_{\mathcal{S}^1}(\mathcal{S}^1 \cap \mathbb{Q})) \cup (C \cap \delta^{-1}_{\mathcal{S}^1}([0,1/2] \cap \mathbb{Q})) \cup (C \cap \delta^{-1}_{\mathcal{S}^1}([-1/2, 0] \setminus \mathbb{Q})$$
$$A^c \cap \mathrm{dom}(\delta_{\mathcal{S}^1}) = (C^c \cap \delta^{-1}_{\mathcal{S}^1}(\mathcal{S}^1 \setminus \mathbb{Q})) \cup (C \cap \delta^{-1}_{\mathcal{S}^1}(]0, 1/2[\setminus \mathbb{Q})) \cup (C \cap \delta^{-1}_{\mathcal{S}^1}(]-1/2, 0[\cap \mathbb{Q})$$

   We show that the piece of information $p \in A$ or $p \in A^c$ is sufficient to computably realize $\tilde{f}$. To do so, let $p \in \mathrm{dom}(\delta_{\mathcal{S}^1})$.

   (a) Suppose that $p \in A$.
   Then it is possible to determine in finite time whether $p \in C$ or $p \in C^c$ (because C and C^c are both effective open sets).
   If $p \in C^c$, then one can continuously associate to p a name p_y or a real number y in $[0, 1[$, where $q(y) = \delta_{\mathcal{S}^1}(p)$ (because $\delta^{-1}_{\mathcal{S}^1}(]-1/8, 1/8[) \subseteq C$), and we return p_y.
   If $p \in C$, then one can continuously associate to p a name p_y of a real number y in $[-1/2, 1/2[$, where $q(y) = \delta_{\mathcal{S}^1}(p)$ (because $\delta^{-1}_{\mathcal{S}^1}(]3/8, 5/8[) \subseteq C^c$). And we return p_y, because: if $y \geq 0$, then $\tilde{f}(\delta_{\mathcal{S}^1}(p)) = y$; and if $y < 0$, then $\tilde{f}(\delta_{\mathcal{S}^1}(p)) = y$.

   (b) Suppose similarly that $p \in A^c \cap \mathrm{dom}(\delta_{\mathcal{S}^1})$.
   If $p \in C^c$, then one can continuously associate to p a real number y in $[0, 1[$, where $q(y) = \delta_{\mathcal{S}^1}(p)$. And we return a name of $y - 1$ (which is computable, because the addition on real numbers is computable).
   If $p \in C$, then one can continuously associate to p a real number y in $[-1/2, 1/2[$, where $q(y) = \delta_{\mathcal{S}^1}(p)$. And one can test in finite time whether $y < 0$ or $y > 0$ (and $y \neq 0$, as all the names of 0 belong in A). If $y > 0$, then we return a name of $y - 1$. And if $y < 0$, then we return a name of $y + 1$.

   Such a procedure realizes $\tilde{f}$, and only depends on the knowing whether $p \in A$ or $p \in A^c$: this is exactly the property 1.

2. However, $\tilde{f}$ is not continuous on any partition of two elements of $\mathcal{S}^1$.

   Suppose by contradiction that A and A^c form such a partition.
   Without loss of generality, we can suppose that $\overline{0} \in A$. As $\tilde{f}(\overline{0}) = 0$, and because $q(\mathbb{Q})$ is dense in $\mathbb{S}^1$, $\tilde{f}$ being continuous imposes it coincides with the identity map on A : this is first true locally in 0, and then step by step on all A. We then conclude that:

$$\lim_{x \nearrow 1^-} \tilde{f_{|A}}(\overline{x}) = 1 \neq 0 = \tilde{f}(\overline{1})$$

   Which is a contradiction!

□

We explain here, between two proofs, why we believe there is a mistake in [Zie12]. The original statement used the representation $q \circ \delta_{[0,1]}$ (where $q : x \mapsto x \mod 1$) and asserts that such a representation is admissible. But we claim it cannot be the case. Indeed, by property C.2.3, if it were we should be able to associate to the sequence of general terms $x_{2n} = \frac{1}{n}$, $x_{2n+1} = 1 - \frac{1}{n}$ (that converges in $\mathcal{S}^1$ towards 0) a converging sequence of names $\mathcal{N}$. But this cannot be the case: the even subsequence converges towards names of 0 (for $\delta_{[0,1]}$), and the odd subsequence converges towards names of 1, which are disjoint.

Finally, the last original result of this section is a generalization of the previous counter-example to any cover of cardinal $k \in \mathbb{N}$:

[Original result] - Example 6.3.4: Piecewise-computability: generalized counter-example

Let $k \in \mathbb{N}, k \geq 2$ and let $p_1, ..., p_{k-1}$ be the first $k-1$ prime numbers. Let $\mathbb{Q}_p$ be the set of rational number whose irreducible form $\frac{n}{d}$ is such that d is a power of p. Define now $f : [0,1] \mapsto \mathbb{R}$ as the following map:

$$\begin{aligned}
f(x) &= x & &\text{if } x \in \mathbb{Q}_{p_1} \cap [0,1) \\
f(x) &= x - 1/k & &\text{if } x \in \mathbb{Q}_{p_2} \cap (0,1) \cup \{1\} \\
&\dots \\
f(x) &= x - (k-1)/k & &\text{if } x \in \mathbb{Q}_{p_{k-1}} \cap (0,1) \\
f(x) &= x - 1 & &\text{if } x \in ([0,1] \setminus (\mathbb{Q}_{p_1} \cup ... \cup (\mathbb{Q}_{p_{k-1}}))
\end{aligned}$$

We map the segment $[0,1]$ to the circle $\mathcal{S}^1$ with the following map: $q : x \mapsto x \mod 1$. There exists a map $\tilde{f} : \mathcal{S}^1 \mapsto \mathbb{R}$ such that $\tilde{f} \circ q = f$. And:

1. There exists k computable maps $F_i :\subseteq \mathcal{N} \mapsto \mathcal{N}$ $(i = 1, ..., k)$ and a partition into k subsets $P_1, ..., P_k \in \Sigma^0_3$ such that:

$$\forall i \in \{1, ..., k\}, \forall p \in P_i \cap \text{dom}(\delta_X), \quad \delta_{\mathbb{R}} \circ F_i(p) = \tilde{f} \circ \delta_{\mathcal{S}^1}(p)$$

2. But there is no partition $\{B_1, ..., B_k\}$ of $\mathcal{S}^1$ that would make f continuous on each of its elements; and *a fortiori*, computable.

Proof.

The proof is very similar to the one of example 6.3.3, however somewhat more laborious. The proof of the impossibility of a continuous partition is exactly identical, so we now focus on the computable k-realizability.

In a similar way than in example 6.3.3, we consider the following two effective open sets $\delta^{-1}_{\mathcal{S}^1}(]-3/8, 3/8[)$ and $\delta^{-1}_{\mathcal{S}^1}(]1/8, 7/8[)$: their intersection is non-empty, and as such it is possible to create an effective clopen (both open and closed) sets C of $\mathcal{N}$ such that $\delta_{\mathcal{S}^1}^{-1}(]-1/8,1/8[) \subseteq C$ and $\delta_{\mathcal{S}^1}^{-1}(]3/8,5/8[) \subseteq C^c$.

We additionally define $A_1 = q([0,1) \cap \mathbb{Q}_{p_1})$, $A_2 = q((0,1] \cap \mathbb{Q}_{p_2})$, ..., $A_{k-1} = q((0,1) \cap \mathbb{Q}_{p_{k-1}})$ and finally $A_k = q((0,1) \setminus (\mathbb{Q}_{p_1} \cup ... \cup \mathbb{Q}_{p_{k-1}})$.

We also define $A_1, ..., A_k$ as:

$$\begin{aligned}
P_1 &= (C^c \cap \delta_{\mathcal{S}^1}^{-1}(A_1)) \cup (C \cap \delta_{\mathcal{S}^1}^{-1}(A_1 \cap [0,1/2])) \cup (C \cap \delta_{\mathcal{S}^1}^{-1}(A_2 \cap (-1/2,0])) \\
P_2 &= (C^c \cap \delta_{\mathcal{S}^1}^{-1}(A_2)) \cup (C \cap \delta_{\mathcal{S}^1}^{-1}(A_2 \cap (0,1/2)\ )) \cup (C \cap \delta_{\mathcal{S}^1}^{-1}(A_3 \cap (-1/2,0)\ )) \\
&\ ... \\
P_{k-1} &= (C^c \cap \delta_{\mathcal{S}^1}^{-1}(A_{k-1})) \cup (C \cap \delta_{\mathcal{S}^1}^{-1}(A_{k-1} \cap (0,1/2)\ )) \cup (C \cap \delta_{\mathcal{S}^1}^{-1}(A_k \cap (-1/2,0)\ )) \\
P_k &= (C^c \cap \delta_{\mathcal{S}^1}^{-1}(A_k)) \cup (C \cap \delta_{\mathcal{S}^1}^{-1}(A_k \cap (0,1/2)\ )) \cup (C \cap \delta_{\mathcal{S}^1}^{-1}(A_1 \cap (-1/2,0)\ ))
\end{aligned}$$

The P_k form a partition of $\delta_{\mathcal{S}^1}^{-1}(\mathcal{S}^1)$ that makes the map $\tilde{f}$ computably realizable on each of its elements. To prove that, we start by determining in finite time whether $p \in C$ or if $p \in C^c$ (as C and C^c are effective open sets). If $p \in C^c$, we can continuously associate to p a name p_y of a real number $y \in [0,1)$. And if $p \in C$, we can continuously associate to p a name p_y of a real number $y \in [-1/2,1/2)$. The rest of the proof is very similar to the manipulations we detailed in the proof of example 6.3.3. □

C.3 Results: the example of real polynomials

C.3.1 Representation of real polynomials and topology

Definition C.3.1: Cauchy representation

Suppose we fixed an enumeration $\{r_n\}_{n\in\mathbb{N}}$ of the set of rational numbers $\mathbb{Q}$. We define the Cauchy representation $\delta_C :\subseteq \mathcal{N} \mapsto \mathbb{R}$ by:

$$\delta_C(p) = x \iff \forall i \in \mathbb{N}, |x - r_{p(i)}| < 2^{-i}$$

Property C.3.2: Admissibility of the Cauchy representation

The Cauchy representation $\delta_C :\subseteq \mathcal{N} \mapsto \mathcal{N}$ is admissible.

Proof. The following proof is inspired by [KW85].

Define for $i, j \in \mathbb{N}$ the open subsets $B_{i,j} = \{x \in \mathbb{R} : |x - r_i| < 2^{-j}\}$. Then $\{B_{i,j}\}_{i,j\in\mathbb{N}}$ is a basis of the euclidean topology on $\mathbb{R}$. Denote by $< \cdot, \cdot >$ an encoding of $\mathbb{N}^2$ into $\mathbb{N}$. We show that the standard representation associated to $\delta_{\mathbb{R}}$ is such that:

$$\text{dom}(\delta_{\mathbb{R}}) = \left\{p \in \mathcal{N} : \exists x \in \mathbb{R}, \{i \in \mathbb{N} : \exists n, p(n) = i + 1\} = \{< i, j >\in \mathbb{N} : d(x, r_i) < 2^{-j}\}\right\}$$
$$\{\delta_{\mathbb{R}}(p)\} = \bigcap_{i,j\in\mathbb{N}:\exists n, p(n)=<i,j>+1} B_{i,j}$$

Now we prove there exists a continuous map $g :\subseteq \mathcal{N} \mapsto \mathcal{N}$ such that $\delta_{\mathcal{R}} = \delta_C \circ g$. To do so, let $p \in \mathcal{N}$. We define a continuous map $\varphi_{p,i} : \mathbb{N} \mapsto \mathbb{N}$, which is an enumeration of $\mathcal{O}_{p,i} = \{< i, j + 1 >\in \mathbb{N} : j \in \mathbb{N}, \exists n, p(n) =< i, j + 1 > +1\}$.
Because of similar reasons from the proof of property 4.1.6, there exists a computable map g such that $g(p)(j)$ is the first value of i (according to the enumeration $\varphi_{p,i}$) such that there exists $n \in \mathbb{N}$, $p(n) =< i, j + 1 > +1$.
And if $p \in \text{dom}(\delta_{\mathbb{R}})$, then on the one hand $p \in \text{dom}(g)$, and on the other hand:

$$\forall j \in \mathbb{N}, |r_{g(p)(j)} - r_{g(p)(j+1)}| \leq |r_{g(p)(j)} - x| + |r_{g(p)(j+1)} - x| < 2^{-j}$$

So we have $\delta_{\mathbb{R}}(p) = \delta_C \circ g(p)$. □

We then build a Cauchy representation δ_C^n on $\mathbb{R}^n$ in the following way:

Definition C.3.3: Cauchy representation δ_C^n

We define the usual encoding $< \cdot, ..., \cdot >: \mathbb{N}^n \mapsto \mathbb{N}$, along with the following encoding $H^{(n)}$ defined recursively by : $H^{(1)}(p) = p$, and

$$H^{(n+1)} : \mathcal{N}^n \mapsto \mathcal{N}$$
$$(p_1, ...p_{n+1})(j) \mapsto \begin{cases} H^{(n)}(p_1, ..., p_n)(y) & \text{if } j = 2y \\ p_{n+1}(y) & \text{if } j = 2y + 1 \end{cases}$$

Then $H^{(n)}$ is an homeomorphism, and we can define the Cauchy representation δ_C^n of $\mathbb{R}^n$ as:

$$\forall p_1, ..., p_n \in \mathcal{N}, \forall j \in \mathbb{N}, \delta_C^n(H^{(n)}(p_1, ..., p_n))(j) =< \delta_C(p_1)(j), ..., \delta_C(p_n)(j) >$$

Such a representation is admissible: see [KW85] for a proof and some other developments. With all of this, we obtained a representation of the spaces $\mathbb{R}_n[X]$ (considered as $\mathbb{R}^{n+1}$). In the following subsections, we demonstrate some properties on $\mathbb{R}[X]$.

Lemma C.3.4: Convergence $\implies$ bound on the degrees

Let $(P_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}[X]$ converging towards an element $P \in \mathbb{R}[X]$. Then there exists some $l \in \mathbb{N}$ such that : $\forall n \in \mathbb{N}, P_n \in \mathbb{R}_l[X]$ and $P \in \mathbb{R}_l[X]$.

Proof. (This statement and its proofs are taken from [Sch02], Lemma 18)

Let $(P_n)_{n\in\mathbb{N}}$ be such a sequence. Suppose that for any $k, l \in \mathbb{N}$, there exists some $n \geq k$ such that $P_n \notin \mathbb{R}_l[X] \cup \{P\}$. With that, we define a map $\varphi : \mathbb{N} \cup \{-1\} \mapsto \mathbb{N} \cup \{-1\}$ by recurrence: $\varphi(-1) = -1$, and $\varphi(i) = \min\{n > \varphi(i-1) : P_n \notin \mathbb{R}_i[X] \cup \{P\}$.

Consider now the set $O = \mathbb{R}[X] \setminus \{P_{\varphi(i)} : i \in \mathbb{N}\}$. For any $i \in \mathbb{N}$, one has a $O \cap \mathbb{R}_i[X] = \mathbb{R}_i[X] \setminus \{P_{\varphi(0)}, ..., P_{\varphi(i)}\}$. This implies $O \cap \mathbb{R}_i[X]$ is an open set in $\mathbb{R}_i[X]$, because any finite subset of a T_1 space is closed. Because of that, O is an open neighborhood of P in $\mathbb{R}[X]$. Because $(P_{\varphi(i)})_{i\in\mathbb{N}}$ is a sequence that converges towards x, there exists $i_0 \in \mathbb{N}$ such that for any $i > i_0$, $P_{\varphi(i)} \in O$. This is absurd, because O does not contain any element of this sequence!

From this, there exists k_0, l_0 such that $\{P_n : n \geq k_0\} \subseteq X_{l_0} \cup \{P\}$. Because $\mathbb{R}_i[X]$ is a subspace of $\mathbb{R}_{i+1}[X]$ for any $i \in \mathbb{N}$, we conclude there exists some $l \in \mathbb{N}$ such that $\{P\} \cup \{P_n : n \in \mathbb{N}\} \subseteq X_l$. ☐

Property C.3.5: $\mathbb{R}[X]$ is not metrizable

$\mathbb{R}[X]$ equipped with the coPolish topology is not metrizable.

Proof. The proof that follows is inspired by [KS05], and is based on the following property:

(L4) : Let $(x_{i,j})_{i,j\in\mathbb{N}}$ be a sequence in X, a first-countable space, such that: for any $i \in \mathbb{N}, \lim_{j\to+\infty} x_{i,j} = x_i$ and such that for any $i \in \mathbb{N}, \lim_{i\to+\infty} x_i = x$. Then there exists two increasing functions φ, ψ, such that: $\lim_{n\to+\infty} x_{\varphi(n),\psi(n)} x$.

Now, define:

$$P_{i,j} = \frac{1}{j} X^i + \frac{1}{i} \quad \text{and} \quad P_i = \frac{1}{i} \quad \text{and} \quad P = 0$$

We temporarily admit this (L4) property, and we will demonstrate it at the end.
The definition 7.1.1 entails that the sequence of general term $P_{i,j}$ converges towards P_i for any $i \in \mathbb{N}$, and that P_i converges towards P. (This is about sequential convergence in $\mathbb{R}_n[X]$, equipped with the euclidean topology, and as such does not raise any surprise).
However, this very same definition 7.1.1 asserts that any sequence of polynomials $(Q_n)_{n\in\mathbb{N}}$ that converges in $\mathbb{R}[X]$ towards Q has bounded degree : this is lemma C.3.4. From there, suppose the (L4) property holds: there would exist two increasing functions φ, ψ such that:

$$\lim_{n\to+\infty} P_{\varphi(n),\psi(n)} = P$$

But the term $P_{\varphi(n),\psi(n)}$ is of degree $\varphi(n)$, and the extracted subsequence does not have a bounded degree, and so cannot converge. This invalidates the L4 property: and because any metric space is first-countable, we conclude that $\mathbb{R}[X]$ is not metrizable.

We now demonstrate the L4 property holds: any reader only interested in the space $\mathbb{R}[X]$ can skip this and jump to the next property.

Let X be a first-countable space, and $(x_{i,j})_{i,j\in\mathbb{N}}$ a double sequence verifying the conditions of the L4 property. Consider $(V_i)_{i\in\mathbb{N}}$ a countable basis of neighborhood of x. We define $W_n = \bigcap_{i=0}^{n} V_i$. $(W_n)_{n\in\mathbb{N}}$, which is still a sequence of neighborhood of x.
As W_0 is an open set that contains p, there exists some $i_0 \in \mathbb{N}$ such that for any $i \geq i_0$, $x_i \in W_0$. Likewise, because the sequence $x_{i_0,j}$ converges towards x_{i_0} which is an element of the open set W_0, there exists some $j_0 \in \mathbb{N}$ such that for any $j \geq j_0$, $x_{i_0,j} \in W_0$. Define $\varphi(0), \psi(0) = i_0, j_0$.
Suppose we have built the $n+1$ first terms $i_0, j_0, ..., i_n, j_n$, of φ and ψ. Likewise, W_{n+1} is an open set containing p, so there exists $i_{n+1} > i_n$ and $j_{n+1} > j_n$ such that for any $i \geq i_{n+1}$, $x_i \in W_{n+1}$ and for any $j \geq j_{n+1}$, $x_{i_{n+1},j} \in W_{n+1}$.
Then, for any neighborhood V of x, there exists some $n_0 \in \mathbb{N}$ such that: for any $n > n_0, x_{\varphi(n),\psi(n)} \in V$. To rephrase this: the sequence of general term $x_{\varphi(n),\psi(n)}$ converges towards x. □

Property 7.1.4: A basis of open sets for $\mathbb{R}[X]$

The coPolish topology on $\mathbb{R}[X]$ has the following basis:

- The open sets of the product topology.
- Along with the sets defined by, for any $h : \mathbb{N} \mapsto \mathbb{R}_+^*$:

$$O_h = \left\{ P = \sum_{k=0}^{\deg(P)} p_k X^k \in \mathbb{R}[X] : \forall j, p_j < h(j) \right\}$$

Proof.

This result comes from some unpublished results by Mathieu Hoyrup. We decompose its proof into an intermediary lemma and a theorem.

Let U_n be a sequence of open sets for the product topology on $\mathbb{R}[X]$ such that $R_n[X] \subseteq U_n$. Then $\bigcap_{n\in\mathbb{N}} U_n$ is an open set of $\mathbb{R}[X]$ equipped with the coPolish topology.

For any $k \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} U_n \cap \mathbb{R}_k[X] = \bigcap_{n<k} U_n \cap \mathbb{R}_k[X]$. This is the intersection of an open set for the product topology with $\mathbb{R}_k[X]$, and as such is an open set for the coPolish topology.

We now prove those "new open sets", along with the open sets of the product topology, form a basis for the coPolish topology.

Let A be an open set for the coPolish topology: there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of open sets of the product topology such that $\forall k \in \mathbb{N}, A \cap \mathbb{R}_k[X] = A_k \cap \mathbb{R}_k[X]$. Without loss of generality, we can suppose that $A_{n+1} \subseteq A_n$ (otherwise, replace A_n by $\bigcup_{k\geq n} A_k$). From there, $A = \bigcap_{n\in\mathbb{N}} A_n$.
Let $x \in A$. There exists $k \in \mathbb{N}$ such that $x \in \mathbb{R}_k[X]$, which means that $x \in A \cap \mathbb{R}_k[X] = A_k \cap \mathbb{R}_k[X] \subseteq A_k$. And because $\mathbb{R}[X]$ equipped with the product topology is regular (or T_3), there exists an open set B of the product topology such that $x \in B \subseteq \overline{B} \subseteq A_k$.
We build a sequence of open sets U_n for the product topology such that $\mathbb{R}_n[X] \subseteq U_n$ and verifying the conditions $x \in B \cap \bigcap_{n\in\mathbb{N}} U_n \subseteq A$, which implies the desired result. More precisely, we build a sequence of sets U_n such that for any n, one has:

$$\overline{B} \cap \mathbb{R}_k[X] \subseteq \overline{B} \cap \overline{U_0} \cap ... \cap \overline{U_{n-1}} \subseteq A_n$$

We first define $U_0 = ... = U_{k-1} = \mathbb{R}[X]$, and the proceed by induction on n. Suppose we already have the previous inclusions. Then:

$$\overline{B} \cap \overline{U_0} \cap ... \cap \overline{U_{n-1}} \cap \mathbb{R}_n[X] \subseteq A_n \cap \mathbb{R}_n[X] \subseteq A \subseteq A_{n+1}$$

And:

$$\mathbb{R}_n[X] \subseteq A_{n+1} \cup \left(R[X] \setminus (\overline{B} \cap \overline{U_0} \cap ... \cap \overline{U_{n-1}})\right)$$

As the left part of this inclusion is compact, and that the left part is an element of the product topology, by regularity there exists an open set for the product topology U_n such that $\mathbb{R}_n[X] \subseteq U_n$ and such that $\overline{U_n}$ is included in the right part of the previous inclusion. To rephrase this:

$$\overline{B} \cap \overline{U_0} \cap ... \cap \overline{U_{n-1}} \cap \overline{U_n} \subseteq A_{n+1}$$

And as the left part of the inclusion contains $\overline{B} \cap \mathbb{R}_k[X]$, we conclude the induction.

We conclude that, for any sequence of real numbers $\varepsilon_n > 0$ that decreases towards 0, the set of polynomials such that, for any n, the coefficient of degree n is lesser than ε_n is an open set for the coPolish topology. And that those sets, along with the open sets for the product topology on the $\mathbb{R}_n[X]$, generate the whole topology. ☐

Property 7.1.3: Representation $\delta_{\mathcal{P}}$

1. $\delta_{\mathcal{P}}$ is an admissible representation of $\mathbb{R}[X]$.
2. The final topology wrt. $\delta_{\mathcal{P}}$ is the coPolish topology.

Proof.

1. The proof that follows is inspired by [Sch02]. First, notice that $\delta_{\mathcal{P}}$ is continuous, because if O is an open set of $\mathbb{R}[X]$, there exists a sequence of open sets $O_n \subseteq \mathbb{R}_n[X]$ such that $O \cap \mathbb{R}_n[X] = O_n$. This leads to:
$$\delta_{\mathcal{P}}^{-1}(O) = \bigcup_{n\in\mathbb{N}} n \cdot (\delta_C^{n+1})^{-1}(O) = \bigcup n \in \mathbb{N} n \cdot (\delta_C^{n+1})(O_n)$$
Which is an open set of $\mathrm{dom}(\delta_{\mathcal{P}})$.

   Furthermore, let $f :\subseteq \mathcal{N} \mapsto \mathbb{R}[X]$ be a continuous map. For any $p \in \mathrm{dom}(f)$, there exists some $n \in \mathbb{N}$ such that $f([p|n]) \subseteq \mathbb{R}_n[X]$. We prove this by contradiction. Indeed, otherwise there would exist a sequence $(P_n)_{n\in\mathbb{N}}$ of $\mathbb{R}[X]$ that would converge towards $f(p)$ by continuity of f, but that would eventually (for n large enough) be in no $\mathbb{R}_l[X]$ for any $l \in \mathbb{N}$: this is a contradiction with lemma C.3.4. So, we can define n_p as being the smallest n verifying this property.
   Let $l \in \mathbb{N}$. Define $f_l :\subseteq \mathcal{N} \mapsto \mathbb{R}_l[X]$ by $f_l(p) = f(p)$ for any $p \in f^{-1}(\mathbb{R}_l[X])$. Then f_l is a continuous map, and because δ_C^{l+1} is admissible, there exists a continuous map $g_l :\subseteq \mathcal{N} \mapsto \mathcal{N}$ such that $f_l = \delta_C^{l+1} \circ g_l$. We can now define:
$$g(p) = n_p \cdot g_{n_p}(p)$$
And then g is continuous because of the choice of n_p, and verifies $f = \delta_{\mathcal{P}} \circ g$.
2. Suppose that $U = \delta_{\mathcal{P}}^{-1}(O)$ is an open set. Then for any $n \in \mathbb{N}$, $U \cap [n]$ is also an open set, and $U \cap [n] = (\delta_C^{n+1})^{-1}(O \cap \mathbb{R}_n[X])$. This means, because of the admissibility of each δ_C^{n+1} on the countably-based space $\mathbb{R}_n[X]$ and of property 4.2.1, that $O \cap \mathbb{R}_n[X]$ is an open set of $\mathbb{R}_n[X]$ for any n. In other words, that O is an open set for the coPolish topology $\mathbb{R}[X]$.

☐

C.3.2 Method of proof - generalities

Property 7.2.2: $\tilde{\Gamma}$-hardness $\implies S \notin \Gamma$

Let X be a topological space, Γ a complexity class such that $\Gamma \neq \tilde{\Gamma}$, and $S \subseteq X$. Then:

$$S \text{ is } \tilde{\Gamma}\text{-hard} \implies S \notin \Gamma$$

Proof.
Let $A \subseteq \mathcal{N}$ be such that $A \in \tilde{\Gamma}(\mathcal{N}) \setminus \Gamma(\mathcal{N})$ (such a subset exists because the hierarchies do not collapse on $\mathcal{N}$: see for example [Kec95], Theorem 22.4).

And let S $\tilde{\Gamma}$-hard. There exists a continuous map $f : \mathcal{N} \mapsto X$ such that $f^{-1}(S) = A$. Suppose by contradiction that $S \in \Gamma$. We obtain that $A = f^{-1}(S) \in \Gamma$: this is absurd! So $S \notin \Gamma$. □

Property C.3.6: Reductions and complexity of preimages

Let X be a topological space equipped with an admissible representation δ and $S \subseteq X, A \subseteq \mathcal{N}$. Suppose there exists a continuous reduction $f : \mathcal{N} \mapsto X$ such that $f^{-1}(S) = A$, and verifying $\delta^{-1}(S) \in \Gamma(\text{dom}(\delta))$, then $A \in \Gamma(\mathcal{N})$.

Proof. Suppose there exists such a reduction. Because δ is admissible, there exists a continuous map $F :\subseteq \mathcal{N} \mapsto \mathcal{N}$ such that: $\forall p \in \text{dom}(f), f(p) = \delta \circ F(p)$. So F is a total map and its codomain is included in $\text{dom}(\delta)$.

From there, $A = f^{-1}(S) = F^{-1}(\delta^{-1}(S)) \in \Gamma(\mathcal{N})$ because F is continuous, which is exactly the desired result. □

Lemma 7.2.3: Belonging in a countable class

Let (X, τ) be a topological space and Γ be a class of the Borel or the difference hierarchy. For any $1 \leq \alpha, \beta < \omega_1$ and $S \subseteq X$:

$$S \in \Gamma(\tau) \iff \exists \tau' \subseteq \tau \text{ countably-based}, S \in \Gamma(\tau')$$

Proof.
Let $\Gamma \in \{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, D_\alpha(\mathbf{\Sigma}^0_\beta)\}$ for any $1 \leq \alpha, \beta < \omega_1$

$\implies$: Let $A \in \Gamma(\tau)$. Writing A involves a finite or countable family of open sets $\{A_i\}_{i \in I}$. By considering τ' the topology induced by this family, τ' is of course countably-based; and one obtains $\tau' \subseteq \tau$ and $A \in \Gamma(\tau')$.

$\impliedby$: Suppose that $A \in \Gamma(\tau')$. As $\tau' \subseteq \tau$, of course $A \in \Gamma(\tau)$. □

Property C.3.7: $\tilde{\Gamma}$-hardness* $\implies S \notin \Gamma$

Let (X, τ) be a topological space, and Γ a complexity class such that $\tilde{\Gamma} \neq \Gamma$ and $S \subseteq X$. Then:

$$S \text{ is } \tilde{\Gamma}\text{-hard}^* \implies S \notin \Gamma$$

Proof.

Suppose S be a $\tilde{\Gamma}(\tau)$-hard* set. Then for any countably-based topology τ' such that $\tau' \subseteq \tau$, S is $\tilde{\Gamma}(\tau')$-hard (in the usual sense as seen in definition 7.2.1). Because of that, by property 7.2.2, $S \notin \Gamma(\tau')$: and by lemma 7.2.3, we conclude that $S \notin \Gamma(\tau)$. ☐

Property C.3.8: A consequence of the $\underset{\sim}{\mathbf{\Delta}}^{\mathbf{0}}_{\mathbf{2}}$-hardness*

Let (X, τ) be a topological space and $S \subseteq X$ a set $\underset{\sim}{\mathbf{\Delta}}^{\mathbf{0}}_{\mathbf{2}}$-hard*. Then:

$$\forall \alpha < \omega_1, S \notin D_\alpha(\underset{\sim}{\mathbf{\Sigma}}^{\mathbf{0}}_{\mathbf{1}})$$

Proof.

Let $S \subseteq X$ be a $\underset{\sim}{\mathbf{\Delta}}^{\mathbf{0}}_{\mathbf{2}}$-hard* subset. We reason by contradiction. Suppose there exists $\alpha < \omega_1$ such that $S \in D_\alpha(\underset{\sim}{\mathbf{\Sigma}}^{\mathbf{0}}_{\mathbf{1}})$. By lemma 7.2.3, there exists a countably-based topology τ' such that $S \in D_\alpha(\underset{\sim}{\mathbf{\Sigma}}^{\mathbf{0}}_{\mathbf{1}})(\tau')$, and S is $\underset{\sim}{\mathbf{\Delta}}^{\mathbf{0}}_{\mathbf{2}}$-hard (for definition 7.2.1) for the topology τ'.

Let α' be an ordinal such that $\alpha < \alpha' < \omega_1$ and let $A \in D_{\alpha'}(\underset{\sim}{\mathbf{\Sigma}}^{\mathbf{0}}_{\mathbf{1}})(\mathcal{N})$. Because $A \in \underset{\sim}{\mathbf{\Delta}}^{\mathbf{0}}_{\mathbf{2}}(\mathcal{N})$, there exists a continuous reduction $f : \mathcal{N} \mapsto X$ such that $f^{-1}(S) = A$ (continuous for the topology τ'). We obtain, as f is continuous, that $A \in D_\alpha(\underset{\sim}{\mathbf{\Sigma}}^{\mathbf{0}}_{\mathbf{1}})(\mathcal{N})$. To rephrase this, the difference hierarchy would collapse at rank α on $\mathcal{N}$: which contradicts exercise 22.26 of [Kec95].

We conclude that for any $\alpha < \omega_1$, $S \notin D_\alpha(\underset{\sim}{\mathbf{\Sigma}}^{\mathbf{0}}_{\mathbf{1}})$. ☐

C.3.3 Method of proof - mathematical foundations

Here is an overview of the method:

1. We exhibit a "canonical" $\underset{\sim}{\mathbf{\Delta}}^{\mathbf{0}}_{\mathbf{2}}$-hard* problem, which we will later always use for reductions. This problem consists in "deciding" (in an unusual sense we call "asymptotic") the value of the limit of a converging sequence with values in $\{0, 1\}$.
2. We then build an algorithmic procedure to create reductions (with lemma 7.2.6) to the problem just above.
3. lemma C.3.9 simplifies the topology we use during this procedure and enables us only to consider the product topology.

We start by introducing our "canonical" $\underset{\sim}{\mathbf{\Delta}}^{\mathbf{0}}_{\mathbf{2}}$-hard* problem:

Property 7.2.5: Converging sequences of $\{0, 1\}^{\mathbb{N}}$

Let $\{0, 1\}^{\mathbb{N}}_{\mathrm{CV}}$ be the set of converging sequences with values in $\{0, 1\}$. A set $A \subseteq \mathcal{N}$ is an element of $\underset{\sim}{\mathbf{\Delta}}^{\mathbf{0}}_{\mathbf{2}}(\mathcal{N})$ if and only if there exists a continuous map $f_A : \mathcal{N} \mapsto \{0, 1\}^{\mathbb{N}}_{\mathrm{CV}}$ such that :

$$\lim_{n \to +\infty} f_A(p)_n = 1 \iff p \in A$$

Proof.

$\implies$: We start by showing that for any $A \in \underset{\sim}{\mathbf{\Sigma}}^{\mathbf{0}}_{\mathbf{2}}(\mathcal{N})$, there exists a continuous map between $\mathcal{N}$ and the space of sequences with values in $\{0, 1\}$ such that $p \in A$ is mapped to a sequence that converges towards 1.

Let $A \in \underset{\sim}{\mathbf{\Sigma}}^{\mathbf{0}}_{\mathbf{2}}(\mathcal{N})$:

$$A = \bigcup_{n \in \mathbb{N}} A_n \setminus A'_n \quad \text{where } A_n, A'_n \in \underset{\sim}{\mathbf{\Sigma}}^{\mathbf{0}}_{\mathbf{1}}$$

We build a Turing machine M with oracle $\mathcal{A}$ that maps $p \in \mathcal{N}$ towards a sequence with values in $\{0, 1\}$, and that converges towards 1 if and only if $p \in A$.

To do so, let $\mathcal{A}$ be the oracle of cylinders $\{\sigma_n \in \mathbb{N}^* : [\sigma_n] \subseteq A_n\}$ and $\{\sigma'_n \in \mathbb{N}^* : [\sigma'_n] \subseteq A'_n\}$. We define M as follows:

1. $M^{\mathcal{A},p}(0) = 0$. The machine creates initializes the variables $L = \emptyset$ and $i = 1$ (i represents the next index of the case it will write in).
2. In this step, at each instant of the computations, the machine writes $M^{\mathcal{A},p}(i) = 0$ and $i \leftarrow i + 1$. M looks in parallel (on all k such that $k \notin L$) in $\mathcal{A}$ for a cylinder $\sigma_k \in \mathbb{N}*$ such that $[\sigma_k] \subseteq A_k$ and that verifies $\sigma_k \sqsubseteq p$. This is a semi-decidable property. If such an index is found, the machine changes $L \to L \cup \{k\}$ and goes to step (3).
3. In this step, at each instant of the computations, the machine writes $M^{\mathcal{A},p}(i) = 1$ and $i \leftarrow i + 1$. M looks in $\mathcal{A}$ for a cylinder $\sigma'_k \in \mathbb{N}*$ such that $[\sigma'_k] \subseteq A'_k$ and that verifies $\sigma'_k \sqsubseteq p$. This is a semi-decidable property. If such an index is found, the machines goes back to step (2).

If $p \in A$, then the sequence $(M^{\mathcal{A},p}(i))_{i\in\mathbb{N}}$ converges towards 1, as there exists $k \in \mathbb{N}$ such that $p \in A_k \setminus A'_k$. Conversely, if the sequence converges towards 1, such an index has been found, and so $p \in A$.

Finally, the map $M^{\mathcal{A}}$ corresponds to a map that is computable relatively to an oracle: by property 3.2.4, this is a continuous map.

Let us now come back to our initial problem. Let $A \subseteq \mathcal{N}$ be an element of $\mathbf{\Delta^0_2}$. Then $A \in \mathbf{\Sigma^0_2}$ and $A^c \in \mathbf{\Sigma^0_2}$. By the previous paragraphs, there exists two machines $M_1^{\mathcal{A}_1}$ and $M_2^{\mathcal{A}_2}$ that associate converging sequences towards 1 to some $p \in \mathbb{N}$ if and only if $p \in A$ (resp. $p \notin A$). We now build a machine $M^{\mathcal{A}_1,\mathcal{A}_2}$, such that:

$$M^{\mathcal{A}_1,\mathcal{A}_2,p}(i) = 1 \quad \text{if} \quad \sum_{j\leq i} M_1^{\mathcal{A}_1}(j) \geq \sum_{j\leq i} M_2^{\mathcal{A}_2}(j)$$
$$0 \quad \text{otherwise}$$

Then the sequence $(M^{\mathcal{A}_1,\mathcal{A}_2,p}(i))_{i\in\mathbb{N}}$ always converges (because one of the two sequences of general terms $M_1^{\mathcal{A}_1}(j)$ and $M_2^{\mathcal{A}_2}(j)$ converges towards 1 by hypothesis), and it converges towards 1 if and only if $p \in A$. Which is the result we wanted.

$\Longleftarrow$: Suppose we have such a map $f_A : \mathcal{N} \mapsto \{0,1\}^{\mathbb{N}}_{\mathrm{CV}}$. In $\{0,1\}^{\mathbb{N}}_{\mathrm{CV}}$, the set W_1 of converging sequences towards 1 is an element of $\mathbf{\Sigma^0_2}$. Indeed, this is a countable set, and as such is written as a union of closed sets. We conclude that $A = f_A^{-1}(W_1)$ is an element of $\mathbf{\Sigma^0_2}$.
Similarly, the set W_0 of converging sequences towards 0 is an element of $\mathbf{\Sigma^0_2}$. So $A^c = f_A^{-1}(W_0)$ has complexity $\mathbf{\Sigma^0_2}$. We conclude that $A \in \mathbf{\Delta^0_2}$. □

We now explain the mathematical foundations of the method. We start by introducing a lemma that simplifies the continuity of the reductions by only considering the product topology:

[Original proof] - Lemma C.3.9: A preliminary lemma

Let $\tau' \subseteq \tau$ be a countably-based topology of $\mathbb{R}[X]$. There exists a countable basis (cf. property 7.1.4 of τ' made of the product topology and of some open sets O_{h_i} defined by some functions $\{h_i\}_{i\in\mathbb{N}}$ O_{h_i} and that form, along with the product topology, a countable basis of τ'.
Define:

$$Y = \left\{ P = \sum_{k=0}^{\deg(P)} p_k X^k \in \mathbb{R}[X] : \forall i, p_i < h_0(i), ..., h_i(i) \right\}$$

Then the induced topology of τ' on Y is exactly the product topology.

Proof.

$\Longrightarrow$: Let O_{h_i} be an open set of τ' generated by a function h_i. Then $O_{h_i} \cap Y$ is an element of the product topology on Y: this open set is generated by the first i coefficients $h_i(0), ..., h_i(i-1)$. And if O is an open set for the product topology on $\mathbb{R}[X]$, then $O \cap Y$ is an open set for the product topology on Y. We conclude the induced topology on Y by τ' is the product topology.

$\Longleftarrow$: By definition of the induced topology, if $O \subseteq Y$ is an open set of Y, there exists some $O' \subseteq \mathbb{R}[X]$ such that O' is an open set of the topology τ'. This means O' is written as a union of elements of the product topology, and of some O_{h_i}, whose intersection with Y form an open set of Y for the product topology by the previous implication. This means O is an element of the product topology on Y. $\square$

We now focus on the theoretical aspects of the method: a "practical" phrasing of it exists in lemma 7.2.6 in the body of this paper. Furthermore, this practical phrasing is only a combination of the previous and following lemma:

[Original proof] - Lemma 7.2.6: A method of reduction for $\mathbf{\Delta}^0_2$

Let $S \subseteq \mathbb{R}[X]$.
Suppose that for any countable-based topology $\tau' \subseteq \tau$ (whose base is made of the product topology and of a countable family of functions $\{h_i\}_{i\in\mathbb{N}}$),
if you define $Y = \{P = \sum_{k\geq 0} p_k X^k \in \mathbb{R}[X] : \forall i, p_i < h_0(i), ..., h_i(i)\}$,
there exists a continuous map (for the product topology) $f_{S,Y} : \{0,1\}^{\mathbb{N}}_{\mathrm{CV}} \mapsto Y$ such that:

$$f_{S,Y}(u) \in S \iff u \text{ converges towards } 1$$

Then S is $\mathbf{\Delta}^0_2$-hard*.

Proof.

Let $\tau' \subseteq \tau$ be a countably-based topology, where τ is the coPolish topology on $\mathbb{R}[X]$. This countably-based topology is generated by a family of functions $\{h_i\}_{i\in\mathbb{N}}$ and by the product topology. We define as in lemma C.3.9:

$$Y = \left\{ P = \sum_{k=0}^{\deg(P)} p_k X^k \in \mathbb{R}[X] : \forall i, p_i < h_0(i), ..., h_i(i) \right\}$$

We show that if there exists a continuous map (for the product topology) $f_{S,Y} : \{0,1\}^{\mathbb{N}}_{\mathrm{CV}} \mapsto Y$ such that:

$$f_{S,Y}(u) \in S \iff u \text{ converges towards } 1$$

then S is $\mathbf{\Delta}^0_2$-hard for the topology τ'. If this holds for any countably-based topology $\tau' \subseteq \tau$, we obtain that S is $\mathbf{\Delta}^0_2$-hard* (for the coPolish topology).

Suppose there exists such a map. The problem of knowing whether a sequence of $\{0,1\}^{\mathbb{N}}_{\mathrm{CV}}$ converges towards 1 or not is a $\mathbf{\Delta}^0_2$-hard problem (cf. property 7.2.5): in other words, if $\{0,1\}^{\mathbb{N}}_{\mathrm{CV1}}$ denotes the subset of $\{0,1\}^{\mathbb{N}}_{\mathrm{CV}}$ of converging sequences towards 1, then for any $B \in \mathbf{\Delta}^0_2(\mathcal{N})$, there exists a continuous map $f_B :\subseteq N \mapsto \{0,1\}^{\mathbb{N}}_{\mathrm{CV}}$ such that $f_B^{-1}(\{0,1\}^{\mathbb{N}}_{\mathrm{CV1}}) = B$

Let $B \in \mathbf{\Delta}^0_2(\mathcal{N})$ and f_B be such an associated map.

Suppose additionally there exists a continuous (for the product topology, and so by lemma C.3.9) continuous for the induced topology) map $f_{S,Y}$ as defined above, then by defining $f = f_{S,Y} \circ f_B$, f is a continuous map that verifies:

$$f^{-1}(S) = f^{-1}(S \cap Y) = f_B^{-1}(f_{S,Y}^{-1}(S \cap Y)) = f_B^{-1}(\{0,1\}^{\mathbb{N}}_{\mathrm{CV1}}) = B$$

With this, S is a $\mathbf{\Delta}^0_2$-hard set for the topology τ'. $\square$

C.3.4 Results: counter-examples

To understand the two demonstrations below (which are additionally original results in themselves), we invite the reader to refer to the section section E, that informally explains this "algorithmic" point of view they use on the difference hierarchy. We also use the method of the previous section to show that a set $S \subseteq X$ is $\boldsymbol{\Delta}^0_2$-hard*: we invite the reader desiring to avoid mathematical technicalities to refer to the informal procedure (available below the lemma 7.2.6), and to temporarily admit the mathematical components behind.

This first counter-example is one of the original results of this internship, although we credit Mathieu Hoyrup for the idea to explore the set of polynomials of even degrees: this is, indeed, a set whose $\boldsymbol{\Delta}^0_2$-hardness* is relatively easy to demonstrate.

[Original result] - Example 7.3.1: A first $\boldsymbol{\Delta}^0_2$-complete* set S_1

Define:

$$S_1 = \{P \in \mathbb{R}[X] : \deg(P) \text{ is even } \}$$

Then S_1 is $\boldsymbol{\Delta}^0_2$-hard*, but $\delta_{\mathcal{P}}^{-1}(S_1) \in D_\omega\left(\boldsymbol{\Sigma}^0_1(\mathrm{dom}(\delta_{\mathcal{P}}))\right)$.

Proof.

1. We first show that $\delta_{\mathcal{P}}^{-1}(S_1) \in D_\omega(\Sigma^0_1(\mathrm{dom}(\delta_P))$. To do so, we create an algorithm A (as in section E) that can announce an integer $k \in \mathbb{N}$ and asymptotically "determine" with at most k mind-changes whether $p \in \delta_{\mathcal{P}}^{-1}(S_1)$.

   Suppose $p \in \mathbb{N}$ is given as input. Then A behaves as follows:

   (a) Reads $p(0)$, announces the integer $k = p(0) + 1$ and initializes $c = -1$.
   (b) Looks for a non-zero coefficient of even degree for $\delta(p)$, and of index strictly greater than c. If it finds one, it switches into state $\mathbf{Y}$ and changes c into the index of this new coefficient.
   (c) Looks then for a non-zero coefficient of odd degree for $\delta(p)$ that is strictly greater than c. If it finds one, it updates c and switches into state $\mathbf{N}$.
   (d) Loops back on 2.

   The fact that c is increasing requires the number of mind-changes to be bounded by k, because $k - 1 = p(0)$ is a bound on the degree of $\delta(p)$. Furthermore, if $p \in \delta_{\mathcal{P}}^{-1}(S_1)$, then the algorithm stabilizes on state $\mathbf{Y}$. Which concludes the proof.

2. We now show that S_1 is $\boldsymbol{\Delta}^0_2$-hard*. To do so, let $h_i : \mathbb{N} \mapsto \mathbb{R}^*_+$ ($i \in \mathbb{N}$) be a countable number of functions and $u \in \{0,1\}^{\mathbb{N}}_{\mathrm{CV}}$ be a converging sequence. We create a reduction as follows:

   (a) Define $P_1 = 1$ if $u_0 = 1$, and $P_1 = 1 + X$ if $u_0 = 0$.
   (b) Browse the sequence $(u_i)_{i\in\mathbb{N}}$ and:
      i. Suppose P_n is the latest polynomial we built. Define now:
      $$\alpha = \min(\frac{1}{2^{n+1}}, h_0(n+1), h_1(n+1), ..., h_{n+1}(n+1))/2$$
      ii. If the sequence transitions from $u_i = 0$ to $u_{i+1} = 1$, define $P_{n+1} = P_n + \alpha X^{n+1}$
      iii. If on the opposite the sequence transitions from $u_i = 1$ to $u_{i+1} = 0$, define $P_{n+1} = P_n + \alpha X^n$.
      iv. Otherwise, nothing happens at this step.

   Then such a sequence converges towards a polynomial $P \in \mathbb{R}[X]$ because the sequence u converges: and P is of even degree if and only if the sequence u stabilized on value 1. Which is exactly the result we wanted.

□

This second example is our own only. The writing of the set may be a bit surprising, but we wanted to underline an interesting resemblance with the proof of the non-metrizability of $\mathbb{R}[X]$ equipped with the coPolish topology: the second to last coefficient contains information about the degree. We only lament this resemblance was found *a posteriori*, and as such could not help in the effort of inventiveness required to obtain such a result:

[Original result] - Example 7.3.2: A second $\utilde{\mathbf{\Delta}}^0_2$-complete* set S_2

Define:

$$S_2 = \left\{ P = \sum_{j=0}^{n-1} \frac{1}{k_j} X^{d_j} + \frac{1}{k_n} X^{d_n} : k_j, d_j \in \mathbb{N}, \forall j, k_j \geq 2k_{j-1} \text{ and } k_{n-1} = d_n \right\}$$

Then S_2 is $\utilde{\mathbf{\Delta}}^{\mathbf{0}}_{\mathbf{2}}$-hard*, but $\delta_{\mathcal{P}}^{-1}(S_2) \in D_2\left(\utilde{\mathbf{\Sigma}}^{\mathbf{0}}_{\mathbf{1}}(\mathrm{dom}(\delta_{\mathcal{P}}))\right)$.

Proof.

1. We first demonstrate that $\delta_{\mathcal{P}}^{-1}(S_2) \in D_2(\Sigma^0_1(\mathrm{dom}(\delta_{\mathcal{P}}))$. To do so, we create an algorithm A (as in section E) of the form **NYN**, which should "decide" with at most two mind-changes whether any $p \in \mathcal{N}$ is or not an element of $\delta_{\mathcal{P}}^{-1}(S_2)$.

   Suppose $p \in \mathbb{N}$ is given as input. Then A behaves as follows:

   (a) Reads $p(0)$ and initializes the variable $k = p(0)$. If $k = 0$, the algorithm loops infinitely on the state **N**. For the sake of clarity, we will use the polynomial $P = \delta_{\mathcal{P}}(p) \in \mathbb{R}_k[X]$ (but it should not make use forget that we are working with a name of P on $\mathcal{N}$). We denote by $(p_i)_{i \leq k}$ the coefficients of the polynomial P, and $p_i^{(t)}$ the approximation given by p of the coefficient p_i after t iterations (and so with a precision at most 2^{-t}).

   (b) The algorithm then computes some $t_0 \in \mathbb{N}$ such that $2^{-t_0} < \frac{1}{k}$, and defines:

   $$B = \{j \in \mathbb{N} : p_j^{(t_0)} > \frac{1}{2k}\}$$

   (c) i. If $\#B = 0$, the algorithm stay in states **N** and loops forever.

   ii. If $\#B = 1$, the algorithm defines $j_1 = \max B$. It supposes that $\frac{1}{p_{j_1}} \in \mathbb{N}$ (which enables to compute in finite time), computes $j_2 = \lfloor \frac{1}{p_{j_1}} \rfloor$ and checks whether $p_{j_2} \neq 0$ (semi-decidable condition). If this is the case, it jumps to step (d) and switches into state **Y**. It defines $\alpha = j_2$, $\beta = j_1$ and $b =$ False.

   iii. If $\#B \geq 2$, the algorithm defines $j_1 = \max B$, $j_0 = \max(B \setminus \{j_1\})$ and checks in finite time (by supposing that $\frac{1}{p_{j_0}}$ is an integer) whether $j_1 = \lfloor \frac{1}{p_{j_0}} \rfloor$.

   A. If this is the case, we define $\alpha = j_1$, $\beta = j_0$, $b =$ True, and switches into state **Y**.

   B. Otherwise, as $\#B = 1$, it computes $j_2 = \lfloor \frac{1}{p_{j_1}} \rfloor$ and checks whether $p_{j_2} \neq 0$ (semi-decidable property). If this is the case, it jumps to step (d) and switches into state **Y**, while it defines $\alpha = j_2$, $\beta = j_1$ and $b =$ False.

   (d) We check in parallel:

   i. Whether there exists some $\alpha' > \alpha$ such that $p_{\alpha'} \neq 0$ (semi-decidable condition). If it finds one, two cases arise:

   A. If $b =$ True, then it checks in finite time by defining $\alpha'' = \lfloor \frac{1}{p_\alpha} \rfloor$ (still by supposing that $\frac{1}{p_\alpha}$ is an integer) whether $\alpha' = \alpha''$. If this is not the case, we switch into state **N**. Otherwise, we stay in state **Y** but updates $\beta \leftarrow \alpha$, $\alpha \leftarrow \alpha'$, and $b \leftarrow$ False. And it jumps to step (d).

   B. Otherwise, it switches to state **N**.

ii. For $(u_i)_{i\in I}$ an increasing enumeration of B, it checks whether there exists some $i \in I$ such that $2p_{u_{i+1}} > p_{u_i}$ or such that $2p_\alpha > p_\beta$ (semi-decidable condition). If it finds one, it switches into state **N**.

iii. If it finds some coefficient whose index is in B (or is α) and that is not the inverse of an integer (semi-decidable condition), or whose index is strictly below β, not in B and non-zero (semi-decidable condition), then it switches into state **N**.

Such an algorithm "asymptotically decides" whether $p \in \delta_{\mathcal{P}}^{-1}(S_2)$ or not.

Suppose that $\delta_{\mathcal{P}}(p) = P \in S_2$. Then there exists some non-zero coefficient of index j' such that $p_{j'}$ is the inverse of an integer, and corresponds to the degree d of the polynomial P. Notice that in this case, $j' \in B$, where B is the set defined above. Two cases arise:

(a) If $d \in B$. Then $d = \max B$ and $j' = \max(B \setminus \{d\})$. The case (c).iii.A is verified, and the algorithm jumps to state (d). Because the polynomial P verifies the good conditions, the algorithm stays in state **Y**.

(b) If $d \notin B$, then there could exist some $j' = \max B$ and $j'' = \max(B \setminus \{j'\})$ such that $\frac{1}{p_{j''}} = j'$, but such that j' is not the degree. But by case (d).i.A the "mistake" in believing that j' was the degree will be eventually corrected, and the conditions of (d) will never be verified: the algorithm stays in state **Y** forever. If $\frac{1}{p_{j''}} \neq j'$, then by the previous case we obtain the right result.

Suppose that $P \notin S_2$. Then some coefficients are not inverse of integers, or do not verify the conditions on the inequalities, and this will be detected in phase (d) and the algorithm will eventually be in state **N**.

Or there is no coefficient whose inverse is the degree of P. Two cases arise again:

(a) The algorithms stays forever in state **N**. Then the "asymptotic" answer is the right one.

(b) Or the algorithms switches in finite time to state **Y**. Then there exists some coefficient whose inverse in the index $j_e \in \mathbb{N}$ of another non-zero coefficient, but that is not the degree. In other words: there exists some non-zero coefficient, of index strictly greater than j'. This is what steps (d).i checks, and it will eventually find such an index: the algorithm will in every case to state **N**.

2. We now show that S_2 is $\utilde{\mathbf{\Delta}}^{\mathbf{0}}_{\mathbf{2}}$-hard*. To do so, let $h_i : \mathbb{N} \mapsto \mathbb{R}_+^*$ ($i \in \mathbb{N}$) be a countable number of functions and $u \in \{0,1\}^{\mathbb{N}}_{\mathrm{CV}}$ be a converging sequence. We create a reduction as follows:

(a) Define $P_1 = 1 + \frac{1}{2}X$ if $u_0 = 1$, and $P_1 = \frac{1}{2} + \frac{1}{4}X$ if $u_0 = 0$.

(b) Browse the sequence $(u_i)_{i\in\mathbb{N}}$ and:

i. Suppose P_n is the latest built polynomial, and let α_n be its dominant coefficient. Define then $d_{n+1} = \frac{1}{\alpha_n}$.

ii. If the sequence transitions from $u_i = 0$ to $u_{i+1} = 1$, define $d_{n+1} = \frac{1}{\alpha_n}$, α_{n+1} be the inverse of an integer such that:

$$\alpha_{n+1} < \min(h_0(d_n), ..., h_{d_n}(d_n), \frac{1}{2^{d_n}}, \alpha_n)/2$$

And then we define $P_{n+1} = P_n + \alpha_{n+1}X^{d_{n+1}}$

iii. If on the opposite the sequence transitions from $u_i = 1$ to $u_{i+1} = 0$, define $d_{n+1} = \frac{1}{\alpha_n}$, α_{n+1} be the inverse of an integer such that:

$$\alpha_{n+1} < \min(h_0(d_n+1), ..., h_{d_n+1}(d_n+1), \frac{1}{2^{d_n+1}}, \alpha_n)/2$$

And then we define $P_{n+1} = P_n + \alpha_{n+1}X^{d_n+1}$.

iv. Otherwise, nothing happens at this step.

Then such a sequence converges towards a polynomial $P \in \mathbb{R}[X]$ because the sequence u converges: and P is an element of S_2 if and only if the sequence u stabilized on value 1. Which is exactly the result we wanted.

□

Lemma 7.3.3: Borel topological and algorithmic complexities coincide

Consider $\mathbb{R}[X]$ equipped with the admissible representation $\delta_{\mathcal{P}}$. For any $S \subseteq \mathbb{R}[X]$ and $1 \leq \beta < \omega_1$,

$$S \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}(\mathbb{R}[X]) \iff \delta_{\mathcal{P}}^{-1}(S) \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}(\text{dom}(\delta_{\mathcal{P}}))$$

Proof.
$\Longrightarrow$: We prove this implication by induction, and use as a main argument the continuity of $\delta_{\mathcal{P}}$. We refer the reader to section C.1.1 for a very analogous proof.

$\Longleftarrow$: Suppose that $\delta_{\mathcal{P}}^{-1}(S) \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}(\text{dom}(\delta_{\mathcal{P}}))$. As $[n] \cap \delta_{\mathcal{P}}^{-1}(S) \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}(\text{dom}(\delta_{\mathcal{P}}))$, by noticing that $[n] \cap \delta_{\mathcal{P}}^{-1}(S) = n \cdot (\delta_C^{n+1})^{-1}(S)$, we obtain that $(\delta_C^{n+1})^{-1}(S) \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}(\text{dom}(\delta_C^{n+1}))$. By applying the corollary over the Borel hierarchy of*theorem* 5.1.1 to each subspace $\mathbb{R}_n[X]$, we only have to show that for any $n \in \mathbb{N}$, if $S \cap \mathbb{R}_n[X] \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}(\mathbb{R}_n[X])$, then $S \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}(\mathbb{R}[X]$. This is the purpose of the following induction over β.

Case $\boldsymbol{\beta = 1}$:
By definition of the coPolish topology, $S \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\mathbf{1}}(\mathbb{R}[X])$ if and only if $S \cap \mathbb{R}_n[X] \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\mathbf{1}}(\mathbb{R}_n[X])$ for every $n \in \mathbb{N}$. The case $\beta = 1$ holds.

Case $\beta > 1$:
Suppose $S \subseteq \mathbb{R}[X]$ is such that for each $n \in \mathbb{N}$, $S \cap \mathbb{R}_n[X] \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}(\mathbb{R}_n[X])$ (and $\beta > 1$).
As for each $n \in \mathbb{N}$, $\mathbb{R}_n[X]$ is a closed set of $\mathbb{R}[X]$, we obtain by applying the properties 4.3.3 and 4.3.4 that $S \cap \mathbb{R}_n[X]$ is an element of complexity $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}(\mathbb{R}[X])$ for each $n \in \mathbb{N}$, because:

$$S = \bigcup_{n \in \mathbb{N}} S \cap \mathbb{R}_n[X] \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\beta}}(\mathbb{R}[X])$$

□

C.3.5 Results: counter-examples to two usual theorems in *DST*

As we announced in the body of this paper, we explore the topological consequences of the two previous results and demonstrate that Hausdorff-Kuratowski and Wadge theorems do no hold on $\mathbb{R}[X]$. Those are the last two original results of this essay:

[Original result] - Theorem 7.4.1: Counter-example to the Hausdorff-Kuratowski theorem

Consider $\mathbb{R}[X]$ equipped with the coPolish topology. There exists a set $S \in \boldsymbol{\Delta}^{\mathbf{0}}_{\mathbf{2}}(\mathbb{R}[X])$ such that:

$$\forall \alpha < \omega_1, S \notin D_\alpha(\boldsymbol{\Sigma}^{\mathbf{0}}_{\mathbf{1}}(\mathbb{R}[X])$$

Proof.
Consider S_1 the set given by example 7.3.1: $S_1 = \{P \in \mathbb{R}[X] : \deg(P) \text{ is even }\}$ Then:

1. $\delta_{\mathcal{P}}^{-1}(S_1) \in D_\omega(\boldsymbol{\Sigma}^{\mathbf{0}}_{\mathbf{1}}(\text{dom}(\delta_{\mathcal{P}})) \subseteq \boldsymbol{\Delta}^{\mathbf{0}}_{\mathbf{2}}(\text{dom}(\delta_{\mathcal{P}}))$. By applying lemma 7.3.3, we obtain that $S_1 \in \boldsymbol{\Delta}^{\mathbf{0}}_{\mathbf{2}}(\mathbb{R}[X])$.
2. By property C.3.8, for any $\alpha < \omega_1$, $S_1 \notin D_\alpha(\boldsymbol{\Sigma}^{\mathbf{0}}_{\mathbf{1}})$.

□

[Original result] - Theorem 7.4.3: Counter-example to the Wadge theorem

Consider $\mathbb{R}[X]$ equipped with the coPolish topology. There exists a set $S \subseteq \mathbb{R}[X]$ such that $S \notin D_2(\boldsymbol{\Sigma}^0_1(\mathbb{R}[X])$, but S is not $\tilde{D}_2(\boldsymbol{\Sigma}^0_1)$-hard (in the usual sense).

Proof.

Define S_2 the set given by example 7.3.2 :

$$S_2 = \left\{ P = \sum_{j=0}^{n} \frac{1}{k_j} X^{d_j} : k_j, d_j \in \mathbb{N}, \forall j, k_j \geq 2k_{j-1} \text{ and } k_{n-1} = d_n \right\}$$

Then:

1. In a similar way to theorem 7.4.1, we have $S_2 \notin D_2(\boldsymbol{\Sigma}^0_1(\mathbb{R}[X])$.
2. Suppose however by contradiction that A is $\tilde{D}_2(\boldsymbol{\Sigma}^0_1)$-hard in the "usual sense". Let now $U \in \tilde{D}_2(\boldsymbol{\Sigma}^0_1(\mathcal{N}))$: there exists a continuous map $f_U : \mathcal{N} \mapsto \mathbb{R}[X]$ such that $f_U^{-1}(A) = U$. As $\delta_{\mathcal{P}}$ is admissible, there exists a continuous map $F : \mathcal{N} \mapsto \mathcal{N}$ such that $\delta_{\mathcal{P}} \circ F = f_U$. In other words, we obtain:
$$U = F^{-1} \circ f_U^{-1}(S_2) \in D_2(\boldsymbol{\Sigma}^0_1(\mathcal{N})$$
This entails that, for any $U \in \tilde{D}_2(\boldsymbol{\Sigma}^0_1(\mathcal{N})$, we have $U \in D_2(\boldsymbol{\Sigma}^0_1(\mathcal{N})$. And the difference hierarchy collapses on $\mathcal{N}$: this is a contradiction to [Kec95], exercise 22.26. So, we conclude that A is not $\tilde{D}_2(\boldsymbol{\Sigma}^0_1)$-hard!

□

Appendix D Computable ordinals

The content of this section are inspired by the following book: Rogers Hartley. *Theory of Recursive Functions and Effective Computability.* MIT Press, Chapter 11, 1987.

In this section we give a brief overview of computable ordinals, and present the main principles of their manipulation. We define here Kleene's system of ordinal notations (cf. definition D.0.2) that we use for our results. Anyone already having knowledge about this can ignore this section, and on the opposite anyone desiring to deepen their understanding of the following notions is invited to read the publication mention above. We suggest the reader to consider what follows as a short explanation sheet giving the key ideas to manipulate computable ordinals as we do in this paper, but nothing more.

We now suppose the reader to be familiar with the definitions of traditional ordinals. We define here "notation systems" on ordinals, which will enable us to do some computational manipulations on them.

In what follows, we fix an enumeration φ of the sequences with values in the set $\{\alpha : \alpha < \omega_1\}$. We use a definition due to Kleene, and define a **notation system** as a function $|\cdot| :\subseteq \mathbb{N} \mapsto \omega_1$ such that:

1. There exists a function k_s such that:

$$|x| = 0 \implies k_s(x) = 0,$$
$$|x| \text{ is a successor } \implies k_s(x) = 1,$$
$$|x| \text{ is a limit ordinal } \implies k_s(x) = 2$$

2. There exists a function p_s such that:

$$|x| \text{ is a successor } \implies |x| = |p_s(x)| + 1$$

3. There exists a function q_s such that:

$$|x| \text{ is a limit ordinal } \implies [(|\varphi_{q_s(x)}(n)|)_{n\in\mathbb{N}} \text{ is an increasing sequence converging towards } |x|]$$

We can now define computable ordinals:

Definition D.0.1: Computable ordinals

1. An ordinal α is **computable** if there exists a notation system assigning at least one notation to α.
2. A notation system is **maximal** if it assigns at least one notation to each computable ordinal.
3. The set of computable ordinals is a segment, and there exists a first non-computable countable ordinal. We denote it $\boldsymbol{\omega_1^{CK}}$ and call it the **Church-Kleene ordinal**.

We now define Kleene's system of ordinal notations:

Definition D.0.2: Kleene's system of ordinal notations

1. **Building:**

   0 receives the notations 1, and we inductively build the other element. Suppose to have build notations for ordinals $< \gamma$, and having defined on them the strict order $<_o$. Then:

   (a) If $\gamma = \beta + 1$, we define the notations of γ as:

   $$\{2^x : x \text{ is a notation of } \beta\}$$

   Similarly, for any 2^x in the set above:

   $$\{z : z <_o 2^x\} = \{x\} \cup \{z : z <_o x\}$$

   (b) If γ is a limit ordinal, we define the notations of γ as:

   $$\{3 \cdot 5^y : (\varphi_y(n))_{n \in \mathbb{N}} \text{ such that } \varphi_y(n) <_o \varphi_y(n+1) \text{ is already in } <_o, \\ \text{and } \gamma \text{ is the limit of the increasing sequence } (|\varphi_y(n)|)_{n \in \mathbb{N}}\}$$

   Similarly, for any $3 \cdot 5^y$ in the set above:

   $$\{z : z <_o 3 \cdot 5^y\} = \{z : \exists n \in \mathbb{N}, z <_o \varphi_y(n)\}$$

2. Furthermore, if O denotes the set of all notations as defined above, we define:

   $$\begin{aligned} k_o(1) &= 0 \\ k_o(2^x) &= 1 \\ k_o(3 \cdot 5^y) &= 2 \\ p_o(2^x) &= x \\ q_o(3 \cdot 5^y) &= y \end{aligned}$$

We use this system as it possesses some interesting properties, such that:

Property D.0.3: Properties of Kleene's system of ordinal notations

1. For any $a \in O$, $\{b : b <_o a\}$ is a « univalent » system: any ordinal strictly inferior to $|a|_O$ possesses a unique notation in this set.
2. For any $a \in O$, the set $\{b : b <_o a\}$ is recursively enumerable in a: in other words, if $\{W_e\}_{e \in \mathbb{N}}$ is a notation of all recursively enumerable subsets of $\mathbb{N}$, there exists a recursive function $f :\subseteq \mathbb{N} \mapsto \mathbb{N}$ such that:
   $$\forall a \in O, W_{f(a)} = \{b \in O : b <_o a\}$$
3. O is a maximal notation system: it associates a notation to any computable ordinal.

We refer the reader interested in deepening its knowledge on computable ordinals to the book mentioned at the beginning of this section.

Appendix E Computable open sets and "algorithms"

In this section, we develop an algorithmic point of view on the difference hierarchy built from the effective open sets, according to some notion of algorithms. This section aims at developing tools "easy to manipulate" for a human being, and will be relatively informal in its statements.

E.1 The case of open sets

Let $(X, \mathcal{B})$ be an effective cb_0 space. While the notion of effective open sets may seem abstract, it is possible to say that effective open sets are semi-decidable properties "in finite time". In other words:

$U \subseteq X$ is an effective open sets if and only if there exists an algorithm such that, if $x \in U$, answers **Y** (Yes) in finite time. If $x \notin U$, its behavior is undefined.

Proof. If such an algorithm exists, it has to answer in finite time, and as such can only analyse any name $p \in \mathcal{N}$ of x in a finite number of points before giving an answer. With this knowledge, one can recursively enumerate a cover of $\delta_X^{-1}(U)$ made of cylinders: which means U is an effective open set.

Conversely, if U is an effective open set, one can recursively enumerate a cover of $\delta_X^{-1}(U)$ made of cylinders. This means that, if $p \in \mathcal{N}$, one has $p \in \delta_X^{-1}(U)$ if an only if it belongs in one of those cylinders: an algorithm returning **Y** when finding such a cylinder prefix of p meets the requirements of the property. □

To generalise this method, we will consider "algorithms" running in infinite time, and disposing of two states of answers **Y** and **N** they can alternate between an "indeterminate" number of times (that depends on the class $D_{(a)}$ of the subset $S \subseteq X$ you consider). This is what we call *mind changes*.
An effective open set is characterized by the existence of an algorithm of the form **NY**: if a point belongs in an effective open set, the algorithm moves from state **N** into state **Y** (which corresponds to one "mind change") ; and if a point doesn't belong in it, the algorithm will stay in state **N** forever.
Similarly, we define algorithms of the form **NYN** that have at most two mind-changes: they can claim that a point belongs in a set D by switching into state **Y** in finite time, and can change its mind an move back later into state **N** from the state **Y**. This answer then has to be fixed. In short: if $p \in D$, then this algorithm has to move into state **Y** and stay in it forever; and if $p \notin D$, the algorithm can stay in state **N**, or effect two transitions $\mathbf{N} \to \mathbf{Y} \to \mathbf{N}$.

E.2 Finite difference of open sets

In the way we could characterize the effective open sets as being subsets/problems described by algorithms of the form **NY** (one mind-change), we show that $D \subseteq X$ is a difference of effective open sets if and only if it is described by an algorithm of the form **NYN** (two mind-changes).

Proof. Let D be a difference of effective open sets, $D = U_1 \setminus U_2$. From two algorithms A_1 and A_2 respectively for U_1 and U_2, we can build an algorithm A for D of the form **NYN**:
Let $x \in X$.

1. If $x \in U_1$, the algorithm A_1 switches to state **Y**. Then A transitions $\mathbf{N} \to \mathbf{Y}$, and goes to step 2. Otherwise, step 1 loops forever.
2. If $x \in U_2$, the algorithm A_2 switches to state **Y**, and then $x \notin D$. So A transitions $\mathbf{Y} \to \mathbf{N}$.

We obtained an algorithm of the form **NYN** that "determines" (in the sense of convergence of sequences in $\{\mathbf{N}, \mathbf{Y}\}$) whether $x \in D$ or $x \notin D$.

Conversely, suppose we have an algorithm of the form **NYN** that describes D. Define U_1 the set of points where the algorithm switches to state **Y**: it is an effective open set. Similarly, define $U_2 \subseteq U_1$ the set of points where the algorithm transitions $\mathbf{Y} \to \mathbf{N}$. Then $D = U_1 \setminus U_2$ is a difference of effective open sets, induced by the **NYN** algorithm we have. □

An induction shows that finite differences of effective open sets are the ones described by algorithms we allow to have a finite number of mind-changes. More formally:

Property E.2.1: "Characterization" of finite differences of open sets

Let $(X, \mathcal{B})$ be an effective cb_0 space and $S \subseteq X$. Then $S \in D_k(\Sigma_1^0)$ (for $k \in \mathbb{N}$) if and only if S is described by an **NYNYN[...]** algorithm with at most k mind-changes.

E.3 Should we go further?

For the sake of curiosity, what happens for countable ordinals in the difference hierarchy?

For levels D_ω, and more generally D_α for any $\alpha \geq \omega$, we could show that a set S is of complexity D_α if and only if it is described by an algorithm of the following type:

1. The algorithm analyses its input x. It can stay in this "undefined state" forever, or eventually declare in finite time an ordinal $\gamma \leq \alpha$.
2. If such an ordinal is declared, the algorithm has to "determine" (always in this asymptotic sense) whether $x \in S$. At each mind-change the algorithm outputs an ordinal such that the sequence of ordinals the algorithm declares at each mind-change is strictly decreasing. If this sequence reaches zero, the algorithm is not allowed any more mind-change.

We invite the reader amused by those considerations to continue this construction for greater ordinals, but we have already covered a lot more than what we will use for the space $\mathbb{R}[X]$.