Prime Compilation of Non-Clausal Formulae

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Joint work with A. Previti, A. Ignatiev and A. Morgado

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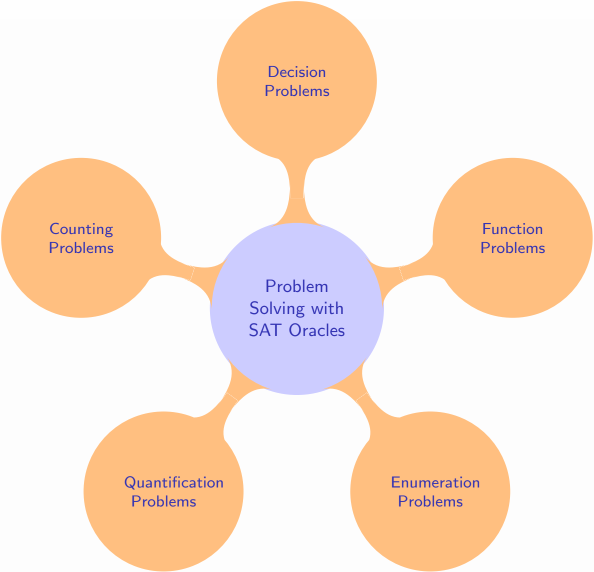
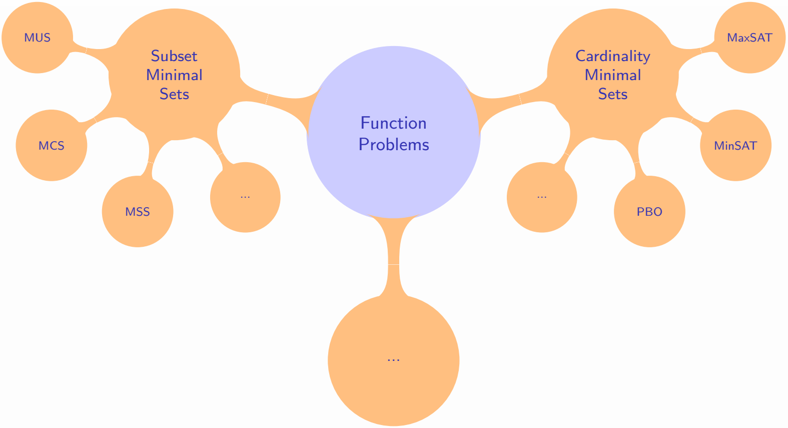
Problem solving with SAT oracles



Function problems



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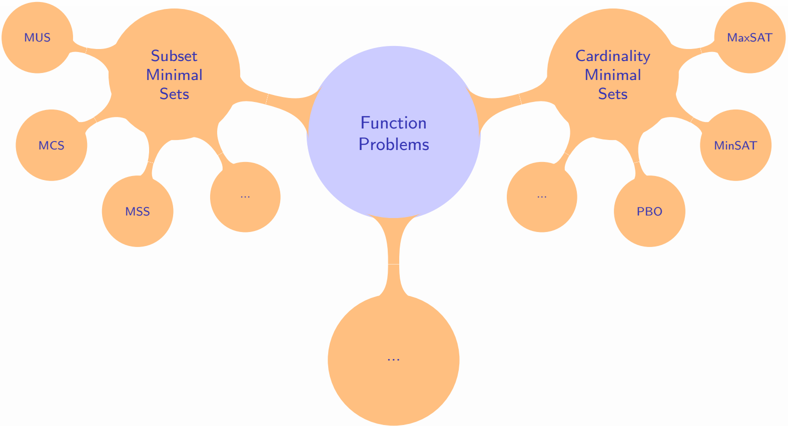

- But also backbones, autarkies, MES, primes, etc.

An example – MUSes

$$(\bar{x}_1 \vee \bar{x}_2) \quad (x_1) \quad (x_5 \vee x_6) \quad (\bar{x}_3 \vee \bar{x}_4) \quad (x_2) \quad (x_3) \quad (x_4)$$

- Formula is unsatisfiable but **not** irreducible

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  - Irreducible subformula that is unsatisfiable
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- Many applications: abstraction in software verification; debugging declarative models; pinpointing in DLs; type error debugging; etc.

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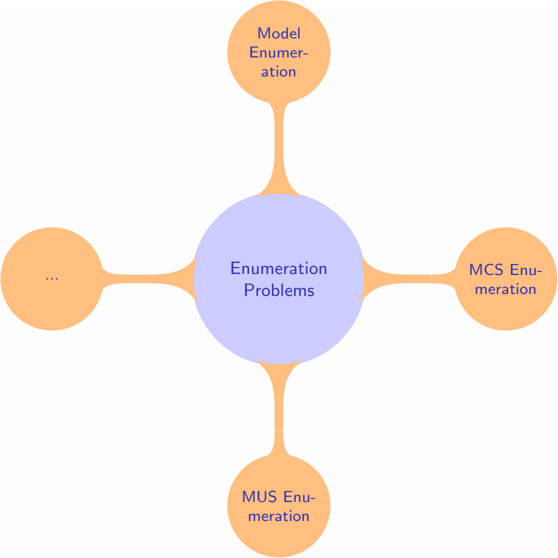
- Many applications: restore consistency; smallest MCSes are MaxSAT solutions; MUS enumeration; minimal/maximal models; etc.

Enumeration problems



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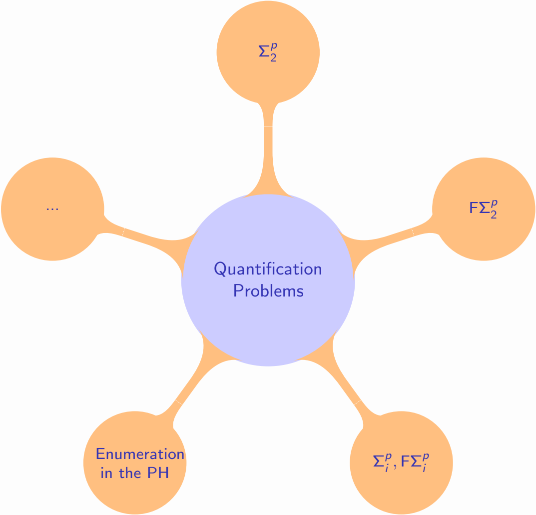
    - Explicit: find all MCSes and dualize
    - Implicit: exploit hitting set dualization and iteratively find MCses and MUSes

Quantification



Application of enumeration – prime compilation

- Enumerate all prime implicates for:

$$(c \lor a) \land (c \lor \neg a) \land (a \lor b \lor d) \land (a \lor b \lor \neg d)$$

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  - Formula minimization; Knowledge compilation; ...

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- **How to enumerate primes of non-clausal formulae, with SAT oracles?**

Outline

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- Non-clausal:
  - Non-CNF and non-DNF
  - Propositional formulae: well-formed formulae built with standard connectives $\neg$, $\wedge$, $\vee$

$$(((a \wedge b) \vee (a \wedge \neg b)) \wedge c) \vee (b \wedge c)$$

Defining primes

- Given formula F, a prime implicate is a non-empty set of non-complementary literals q, s.t.

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- **Each prime implicant (resp. implicate) of F is a minimal hitting set of the prime implicates (resp. implicants) of F** [R94]

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- **How about the general case of prime implicates for CNF, prime implicants for DNF, or primes for non-clausal?**
- **And, how about enumeration of primes?**
  - Repeated application of procedure above does not work...

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  - An MCS C is the complement (wrt to F) of an MSS S, $C = F \setminus S$
  - **Each MCS (resp. MUS) of F is a minimal hitting set of the MUSes (resp. MCSes) of F** [R'87,BL'03,BS'05,LS'08]

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- Group MUS, $\Psi \subseteq \Gamma$:

$$\bigwedge_{\substack{G_i \in G_0 \cup \Psi \\ c \in G_i}} (c) \vDash \perp \;\wedge\; \forall_{\Psi' \subsetneq \Psi} \bigwedge_{\substack{G_i \in G_0 \cup \Psi' \\ c \in G_i}} (c) \nvDash \perp$$

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- **Note:** F is a (possibly non-clausal) propositional formula

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- **How to compute group MUSes?**

Extracting MUSes

- Many algorithms, based on calls to SAT oracles:
  - Deletion-based [CD91,BDTW93]
  - QuickXplain [J04]
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An example

$$F = (c \lor a) \land (c \lor \neg a) \land (a \lor b \lor d) \land (a \lor b \lor \neg d)$$

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  - $\{ \text{c} \}$ is a prime implicate of F, i.e. $F \vDash c$

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- Use B (initially $B = \emptyset$) to block computed prime implicants
  - $H = L \cup C \cup B$

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  - No more (minimal) models

Other approaches

- Clausal formulae:
  - Problem reformulation
    - See above, but restricted
  - Iterated consensus/resolution, since the 1950s
  - Use of BDDs
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Outline

Background

Related Work

Primes for Non-Clausal Formulae

Results

An example

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- **How to enumerate primes of non-clausal formulae, with SAT oracles?**

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- **Idea**: Construct C on demand as the algorithm executes; terminate when B blocks all primes **and** C equivalent to F

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- Update H and repeat

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input : Formula F
output: $PI_n(F)$ and prime implicate cover of F

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while true **do**
  $(\mathrm{st}, A^H) \leftarrow \mathrm{MaxModel}(H)$
  if not st **then return**

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    $(\mathrm{st}, A^H) \leftarrow \mathrm{MaxModel}(H)$
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    $A^F \leftarrow \mathrm{Map}(A^H)$ `# Generate assignment for` F
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Algorithm 1

input : Formula F
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```
H ← {(¬x_v ∨ ¬x_¬v) | v ∈ var(F)}        # Initially, C = ∅ and B = ∅
while true do
    (st, A^H) ← MaxModel(H)
    if not st then return
    A^F ← Map(A^H)                        # Generate assignment for F
    st ← SAT(A^F ∪ ¬F)
    if not st then                        # A^F ⊨ F; i.e. A^F is an implicant
        I_n ← ReduceImplicant(A^F, F)
        ReportPrimeImplicant(I_n)
        b ← {¬x_l | l ∈ I_n}              # Update B by blocking prime implicant
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    $b \leftarrow \{x_l \mid l \in I_e\}$ # Update C by blocking prime implicate

  $H \leftarrow H \cup \{b\}$

Example for algorithm 1

$$H = L \cup B \cup C$$

$$F = (((a \wedge b) \vee (a \wedge \neg b)) \wedge c) \vee (b \wedge c)$$

- SAT oracle query: $F \wedge A^F$

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Non-clausal prime compilation – approach 2

- Iteratively compute **minimal** models A^H of working formula H
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- Update H and repeat

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        $I_n \leftarrow A^F$
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Outline

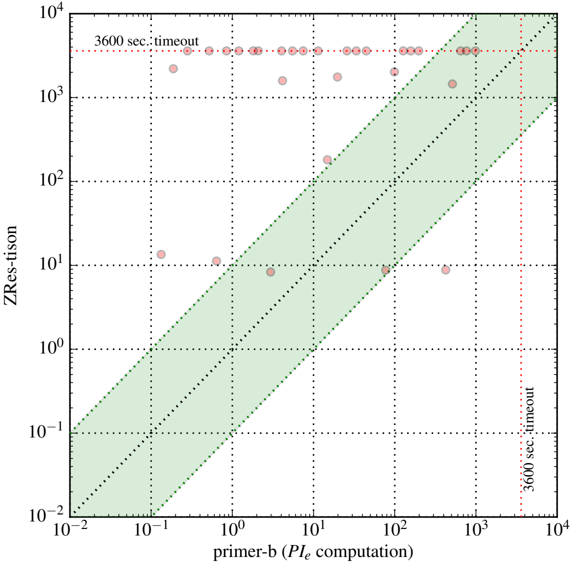
Experimental setup

- Server: Intel Xeon E5-2630 2.60GHz, 64GByte
- TO: 3600s
- MO: 10 GByte
- Tools:
  - primer: PRIMe compilER
  - zres-tison [SdV01]
- Benchmarks:
  - Quasigroup classification problems: 83
  - Cryptanalysis of the Geffe stream generator: 600
  - Crafted $F_m \vee PHP_n$: 30
    - $F_m = (x_1 \vee y_1) \wedge \cdots \wedge (x_m \vee y_m)$
    - $m \in \{10, \ldots, 20\}$
    - $n \in \{6, \ldots, 10\}$
  - Crafted $F_m \vee GT_n$: 30
    - $n \in \{12, \ldots, 20\}$

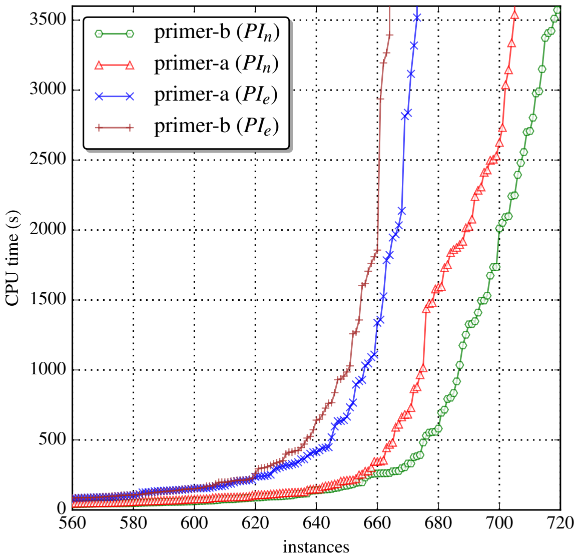
Summary of results

	QG6	Geffe gen.	F+PHP	F+GT	Total
# instances	83	600	30	30	743
ZRes-tison	0	0	11	0	11
primer-a (PI_n)	53	**596**	**30**	26	705
primer-a (PI_e)	28	588	**30**	27	673
primer-b (PI_n)	**64**	595	**30**	**30**	**719**
primer-b (PI_e)	30	577	**30**	27	664

F+PHP scatter plot



Comparing algorithms



Conclusions & future work

- Enumeration of prime implicants for non-clausal formulae with SAT oracles

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- Improvements to proposed algorithms
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- Other compilation languages?

Thank You