A measured approach towards “good SAT representations”

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Clause-sets

- Let $\mathcal{VA}$ be the set of variables.
- Let $\mathcal{LIT}$ be the set of literals, which are either variables or complemented variables, i.e., $\mathcal{LIT} = \mathcal{VA} \mathbin{\dot\cup} \overline{\mathcal{VA}}$.
- A clause is a finite and complement-free subset of $\mathcal{LIT}$, the set of all clauses is $\mathcal{CL}$.
- Let $\mathcal{CLS}$ be the set of clause-sets, finite subsets of $\mathcal{CL}$.

$$\begin{aligned} \bot &:= \emptyset \in \mathcal{CL} \\ \top &:= \emptyset \in \mathcal{CLS}. \end{aligned}$$

SAT Knowledge Compilation

We have only a very scant understanding of "SAT encoding".
These are fragments of a theory.

$$\begin{aligned} \mathrm{hd} &: \mathcal{CLS} \to \mathbb{N}_0 \\ \mathrm{phd} &: \mathcal{CLS} \to \mathbb{N}_0 \\ \mathrm{awid} &: \mathcal{CLS} \to \mathbb{N}_0. \end{aligned}$$

"Hardness" for historical reasons; $\mathrm{hd} = \mathrm{thd}$.

A Framework

$\mathrm{hd}, \mathrm{phd}, \mathrm{awid}$ are **Target-Parameters** for "SAT KC":

1. "Hardness" concerns very simple, oblivious SAT algorithms.
2. SAT-measurement by worst-case from UNSAT.
3. UNSAT-measurements as stable versions of resolution complexity.

What's the SAT solver to do?

The idea of

$$\mathrm{hd}(F) = k, \mathrm{phd}(F) = k$$

resp.

$$\mathrm{awid}(F) = k$$

is:

With a generic, oblivious algorithm using time $n^{O(k)}$
and space $n^{O(1)}$ resp. $n^{O(k)}$
all "implicit information" of F can be uncovered.

k is a structural parameter of F, measuring at which maximal "level" we can extract prime implicates from F.

That "extraction" is implicitly and partially done
by the SAT solver, who makes the "queries".

Resolution efforts

We have $\text{hd}(F) \leq k$ resp. $\text{awid}(F) \leq k$ iff for all prime implicates C of F there is a resolution derivation of C from F such that

from all nodes there exists a path to some leaf of length at most k

resp.

after removal of the literals of C from the derivation,
for every resolution step at least one of the parent clauses
has length at most k.

Examples for the audience: $k = 0, 1$.

Hierarchies

For $k \in \mathbb{N}_0$:

$$\begin{aligned} \boldsymbol{\mathcal{UC}_k} &:= \{F \in \mathcal{CLS} : \mathrm{hd}(F) \le k\} \\ \boldsymbol{\mathcal{PC}_k} &:= \{F \in \mathcal{CLS} : \mathrm{phd}(F) \le k\} \\ \boldsymbol{\mathcal{WC}_k} &:= \{F \in \mathcal{CLS} : \mathrm{awid}(F) \le k\}. \end{aligned}$$

$\mathcal{WC}_0 = \mathcal{UC}_0$: clause-sets which contain all their prime implicates.
$\mathcal{UC} := \mathcal{UC}_1 = \mathcal{WC}_1$ showed up in two different contexts:

1. $\mathcal{UC}$ was introduced in del Val [6] for the purpose of Knowledge Compilation (KC).
2. In [7, 9] we showed $\mathcal{UC} = \mathcal{SLUR}$, continuing Čepek, Kučera, and Vlček [5], for the umbrella class $\mathcal{SLUR}$ for polytime SAT decision as introduced in Schlipf, Annexstein, Franco, and Swaminathan [15].

More generally we have $\mathcal{UC}_k = \mathcal{SLUR}_k$ for $k \ge 0$.

Propagation hardness

$\mathcal{PC} := \mathcal{PC}_1$ was introduced by Bordeaux and Marques-Silva [4].
We have

$$\mathcal{PC}_0 \subset \mathcal{UC}_0 \subset \mathcal{PC}_1 \subset \mathcal{UC}_1 \subset \mathcal{PC}_2 \subset \mathcal{UC}_2 \dots$$

We introduced the $\mathcal{PC}_k$ classes in [10, 11]. Roughly:

phd$(F) = k$ refines hd$(F) = k$
by a strengthened derivation condition —
prime implicates must be derivable by weaker means
(which can not be given by the geometry of the resolution refutation).

Outline

From USAT to SAT

- Let $\mathcal{USAT} := \mathcal{CLS} \setminus \mathcal{SAT}$.
- Let $\mathcal{PASS}$ be the set of partial assignments.
- For $\varphi \in \mathcal{PASS}$ and $F \in \mathcal{CLS}$ let $\varphi * F \in \mathcal{CLS}$ be the result of applying φ to F.

In Beyersdorff and Kullmann [3] the following approach was formally introduced:

Consider $h_0 : \mathcal{USAT} \to \mathbb{N}_0$.

We extend to $h : \mathcal{CLS} \to \mathbb{N}_0$ by

$$h(F) := \max\{h_0(\varphi * F) : \varphi \in \mathcal{PASS} \wedge \varphi * F \in \mathcal{USAT}\}.$$

If we assume that applying partial assignments does no increase h_0 (and this we always do), then this holds also for h.

Many characterisations of hardness's I

We characterise $\mathrm{hd}(F)$ and $\mathrm{awid}(F)$ (indeed for arbitrary $F \in \mathcal{CLS}$) by games in [3], extending

- Pudlák and Impagliazzo [14]
- and Atserias and Dalmau [1].

Since the hardness-game can be simulated by the asymmetric-width game, we obtain

$$\forall F \in \mathcal{CLS} : \mathrm{awid}(F) \leq \mathrm{hd}(F).$$

Algorithmically appealing are the characterisations of $\mathrm{hd}, \mathrm{phd}$ via generalised UCP.

Generalised UCP

Let $\mathrm{r}_k : \mathcal{CLS} \to \mathcal{CLS}$ denote generalised unit-clause propagation.

- r_1 is UCP.
- r_2 is (complete) failed literal elimination.

Now for $F \in \mathcal{USAT}$:

$$\mathrm{hd}(F) = \min\{k \in \mathbb{N}_0 : \mathrm{r}_k(F) = \{\bot\}\}$$

So $\mathrm{hd}(F)$ is the minimal level where r_k detects unsatisfiability. Via the general extension follows for $F \in \mathcal{CLS}$:

$$\mathrm{hd}(F) = \min\{k \in \mathbb{N}_0 \mid \forall \varphi \in \mathcal{PASS} : \varphi * F \in \mathcal{USAT} \Rightarrow \mathrm{r}_k(F) = \{\bot\}\}.$$

Characterising p-hardness

phd on $\mathcal{USAT}$ is just hd, so this special measure is not defined by the general extension process.

Instead we have for $F \in \mathcal{CLS}$:

$$\mathrm{phd}(F) = \min\{k \in \mathbb{N}_0 \mid \forall\, \varphi \in \mathcal{PASS} : \mathrm{r}_k(\varphi * F) = \mathrm{r}_\infty(F)\},$$

where $\mathrm{r}_\infty : \mathcal{CLS} \to \mathcal{CLS}$ is the complete elimination of forced literals (forced assignments, implied units, backbone literals).

Relations to resolution complexity

For $F \in \mathcal{USAT}$ holds:

$$2^{\mathrm{hd}(F)} \leq \mathrm{Comp}_{\mathrm{R}}^{*}(F) \leq (n(F)+1)^{\mathrm{hd}(F)}$$

$$\exp\left(\frac{1}{8}\frac{\mathrm{awid}(F)^2}{n(F)}\right) < \mathrm{Comp}_{\mathrm{R}}(F) < 6 \cdot n(F)^{\mathrm{awid}(F)+2}$$

where

- $\mathrm{Comp}_{\mathrm{R}}^{*}(F)$ is the minimal number of leaves in a tree resolution refutation of F;
- $\mathrm{Comp}_{\mathrm{R}}(F)$ is the minimal number of nodes in a dag resolution refutation of F.

Basic relations

$$\mathcal{PC}_0 \subset \mathcal{UC}_0 \subset \mathcal{PC}_1 \subset \mathcal{UC}_1 \subset \mathcal{PC}_2 \subset \mathcal{UC}_2 \dots$$

$$\mathcal{WC}_0 \subset \mathcal{WC}_1 \subset \mathcal{WC}_2 \subset \dots$$

$$\mathcal{UC}_0 = \mathcal{WC}_0$$

$$\mathcal{UC}_1 = \mathcal{WC}_1$$

$$\mathcal{UC}_k \subset \mathcal{WC}_k \text{ for } k \geq 2$$

$$\mathcal{PC}_{k+1} \not\subseteq \mathcal{WC}_k \text{ for } k \geq 0$$

$$\mathcal{WC}_3 \not\subseteq \mathcal{UC}_k \text{ for } k \geq 0.$$

Open Problem

For the last relation, can we use $\mathcal{WC}_2$?

Decision complexity

$$\mathcal{PC}_0 = \{\top\} \cup \{F \in \mathcal{CLS} : \bot \in F\}.$$

($\mathcal{PC}_0$ is the only functionally incomplete level.)

$\mathcal{UC}_0 = \mathcal{WC}_0$ is decidable in polynomial time.
(These are the primal clause-sets (modulo subsumption).)

All $\mathcal{UC}_k, \mathcal{PC}_k, \mathcal{WC}_k$ for $k \geq 1$ are coNP-complete.
(Via simple reductions to the first level, applying Čepek et al. [5] (SLUR) and Babka, Balyo, Čepek, Štefan Gurský, Kučera, and Vlček [2].)

Strong separation

In Gwynne and Kullmann [8] we show:

Theorem

For all $k \geq 0$ there are (sequences of) short clause-sets in $\mathcal{UC}_{k+1}$, where all (sequences of) equivalent clause-sets in $\mathcal{WC}_k$ are of exponential size.

Conjecture

This strong separation holds between classes $\mathcal{C}, \mathcal{D} \in \{\mathcal{UC}_p, \mathcal{PC}_p, \mathcal{WC}_q\}$ iff it is not trivially false, i.e., iff $\mathcal{C} \not\subseteq \mathcal{D}$.

Allowing auxiliary variables

Consider $F, G \in \mathcal{CLS}$ with $\mathrm{var}(F) \subseteq \mathrm{var}(G)$.

Definition

*G **represents** F if the satisfying assignments of G projected to $\mathrm{var}(F)$ are precisely the satisfying assignments of F.*

Conjecture

For all $k \geq 0$ there are (sequences of) short clause-sets in $\mathcal{UC}_{k+1}$, where all (sequences of) representing clause-sets in $\mathcal{WC}_k$ are of exponential size.
More generally, such a separation holds between classes $\mathcal{C}, \mathcal{D} \in \{\mathcal{UC}_p, \mathit{Propc}_q, \mathcal{WC}_q\}$ iff it is not trivially false.

The "relative condition"

If G represents F, then the **absolute condition** for G is a requirement

- $G \in \mathcal{UC}_k$ or
- $G \in \mathcal{WC}_k$

for some suitable k.

So the requirements on prime implicates also concern
prime implicates containing auxiliary variables
(i.e., variables in G but not in F).

Now the **relative condition** considers only prime implicates with variables from F.

We then speak of **relative hardness**.

This is, when using auxiliary variables, a weaker requirement.

Collapse under the relative condition

In [13] we show:

Theorem

Allowing representations with auxiliary variables, under the relative condition all classes $\mathcal{UC}_k, \mathcal{PC}_k, \mathcal{WC}_k$ *collapse in polynomial time to* $\mathcal{UC}_0$ *or* $\mathcal{PC}_1$.

"Relative $\mathcal{PC}_1$" is indeed what nearly everybody uses for SAT representations, typically called "generalised arc-consistency".

Conjecture

There are (sequences of) clause-sets which have short representations of relative hardness 1*, but for each* k *have only (sequences of) superpolynomial / exponential size representations in* $\mathcal{WC}_k$.

Strongly forcing

Theorem ([11])

From a family of clause-sets F and $V \subseteq \mathrm{var}(F)$, such that the relative asymmetric width of F w.r.t. V is a constant k, we can compute in polynomial time a $G \in \mathcal{CLS}$ with $V \subseteq \mathrm{var}(G)$ such that

- *G represents the same boolean function w.r.t. V as F.*
- *G has relative p-hardness 1.*
- *Moreover, for every φ with $\mathrm{var}(\varphi) = V$, such that $\varphi * G$ is satisfiable, running unit-clause propagation on $\varphi * G$ yields $\top$.*

The terminology “strongly forcing” has been developed in collaboration with Donald Knuth (for his forthcoming fascicle on satisfiability).

Summary and outlook

I Hopefully a theory of “good SAT representations” will emerge.

II The translation of XOR-systems is a good first test-case: Despite the bad news “no poly-size good representation” ([10, 11]), there seem to be a lot of opportunities for good representations (under various circumstances).

III Fascinating connections to space-measurements for resolution (which also yield target classes!).

IV By [12]: For $F \in \mathcal{CLS}$ holds $\mathrm{wid}(F) \le \mathrm{tw}(F) + 1$ (symmetric width vs. primal treewidth). We believe the Conjecture ([11]): $\mathrm{awid}(F) \le \mathrm{tw}^*(F)$ (asymmetric width vs. incidence treewidth).

End

(references on the remaining slides).

For my papers see
http://cs.swan.ac.uk/~csoliver/papers.html.

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