Complexity aspects of CNF to CNF compilation

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CNF to CNF compilation

- **Input**: arbitrary CNF
- **Output**: logically equivalent CNF with “good” inference properties
- **Compilation method**: add implicates
- **What do we mean by “good” inference properties?**
“Good” inference properties

- In general – polynomial time procedure to
  - prove that a given clause is an implicate of the given CNF (clausal entailment)
  - discover all entailed literals after any partial substitution

- In our context – we want unit propagation (UP) to suffice for both tasks
Clausal entailment by UP

- Clause $C$ is 1-provable w.r.t. CNF formula $F$ iff unit propagation on $F \land \neg C$ derives $\bot$.
- CNF $F$ is unit refutation complete if every implicate of $F$ is 1-provable w.r.t. $F$.
  (definition due to Alvaro del Val 1994)
- URC = class of unit refutation complete CNFs
Literal entailment by UP

- CNF $F$ is **propagation complete** if for any partial assignment $x_1, \ldots, x_k$ and any literal $d$: if $d$ is entailed from $F$ by $x_1, \ldots, x_k$ then $d$ is also entailed by unit propagation on $F$ after fixing the values of $x_1, \ldots, x_k$. (definition due to Lucas Bordeaux, Joao Marques-Silva 2012)

- PC = class of propagation complete CNFs
URC versus PC

- Easy to see $PC \subseteq URC$ but not vice versa
  
  $$F = (a \lor b \lor x) \land (a \lor c \lor \neg x)$$

- $F$ is not in $PC$ ($b=0$ and $c=0$ entail $a=1$ but unit propagation does not discover this fact)

- $F$ is in $URC$ (the only prime implicate not explicitly present in $F$ is $a \lor b \lor c$ and it is clearly 1-provable)
How to achieve PC?

- Which clauses are worth adding?
- Clause $C = x_1 \lor \ldots \lor x_k$ is an empowering implicate for CNF $F$ if $C$ (after a possible renumbering) contains an empowered literal $x_k$ such that
  - $F \land \neg x_1 \land \ldots \land \neg x_{k-1}$ entails $x_k$
  - unit propagation run on $F \land \neg x_1 \land \ldots \land \neg x_{k-1}$ entails neither $\bot$ nor $x_k$

(definition due to Darwiche, Pipatsrisawat 2011)
Notes on empowering implicates

- Asserting clauses learnt by CDCL SAT solvers are empowering implicates for the “current” CNF held by the solver.
- CNF \( F \) is propagation complete iff there exists no empowering implicate for \( F \).
- CNF \( F \) can be turned into a propagation complete CNF by repeatedly adding empowering implicates (compilation process).
Complexity issues of such KC

- Given CNF $F$ and clause $C$ what is the complexity of deciding whether $C$ is an empowering implicate for $F$?
- Given CNF $F$ what is the complexity of deciding whether there exists an empowering implicate for $F$?
- Given CNF $F$ how many empowering implicates need to be added to achieve PC?
Result 1

- **Question**: Given CNF $F$ and clause $C$ what is the complexity of deciding whether $C$ is an empowering implicate for $F$?

- **Answer**: The problem is co-NP-complete.
Result 2

- **Question**: Given CNF $F$ what is the complexity of deciding whether there exists an empowering implicate for $F$?
- **Answer**: The problem is NP-complete.
- **Corollary**: Testing whether a given CNF is PC is co-NP-complete.
Result 3

- **Question**: Given CNF $F$ how many empowering implicates need to be added to achieve propagation completeness?

- **Answer**: There are CNFs for which an exponential number of empowering clauses has to be added to arrive to a PC formula.
Connection to CSP

- In CSP each variable $X_i$ has its finite domain $D(X_i)$.
- Constraint – specifies which combinations of values from domains are allowed.
- Propagator $P$ for a constraint $C$ – an algorithm that restricts the domains of variables appearing in $C$
- $P$ detects dis-entailment $\iff P$ returns an empty domain whenever $C$ has no solution
- $P$ enforces domain consistency $\iff$ for every domain value $d \in D(X_i)$ returned by $P$, there is a solution of $C$ in which $X_i = d$. 
Binarization of CSP variables

- Direct encoding – one Boolean variable for every value in every domain
  \[ x_{ij} = 1 \iff X_i = j \text{ for } j \in D(X_i) \]

- ALO clauses
  \[ \forall i : (x_{i1} \lor x_{i2} \lor \ldots \lor x_{ip}) \]

- AMO clauses
  \[ \forall i \forall j \neq k : (\neg x_{ij} \lor \neg x_{ik}) \]
CNF decomposition for a propagator

- CNF decomposition $F_P$ for a propagator $P$ is a CNF on variables $(x, y)$ where $x$ is the set of variables from the direct encoding and $y$ is a set of auxiliary variables, such that
  - UP derives $\bot$ on $F_P$ $\iff$ $P$ returns an empty domain
  - $x_{ij} \leftarrow 0$ by UP on $F_P$ $\iff$ $P$ removes $j$ from $D(X_i)$
- $P$ detects dis-entailment $\iff$ $F_P$ is URC
- $P$ enforces domain consistency $\iff$ $F_P$ is PC
Open problem

- What is the gap between the size of the input and the PC output if both CNFs are compiled into some other representation (e.g. ZBDD)?
Thank you for your attention.