A Strongly Exponential Separation of DNNFs from CNFs

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joint work with
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ALGORITHMS AND COMPLEXITY GROUP

Outline

Motivation

Contribution

Proof

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Representation Languages

In choosing a *representation language* for a propositional theory there is a trade-off between "succinctness" and "tractability".

Darwiche and Marquis (2002) systematically investigate a hierarchy of representation languages that strike this balance in different ways.

Representation Languages

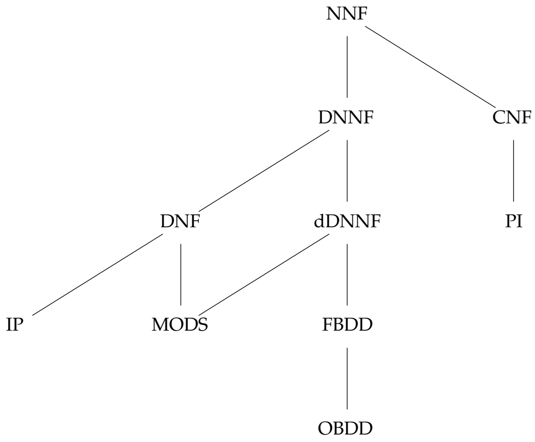

Figure: Inclusion relation on representation languages (Hasse diagram).

Representation Languages

Negation Normal Forms (NNF) Boolean circuits having unbounded fanin AND and OR gates with negations pushed to the input gates.

Decomposable NNFs (DNNF) NNFs where subcircuits leading into each AND gate are defined on disjoint sets of variables.

Deterministic DNNFs (dDNNF) DNNFs where subcircuits leading into each OR gate never simulataneously evaluate to 1.

Conjunctive Normal Forms (CNF) NNFs where...

Prime Implicate Forms (PI) CNFs where entailed clauses are already entailed by a single clause in the CNF and no clause in the CNF is entailed by another.

... ...

size(C) is the number of arcs in the DAG underlying C (for C in NNF).

Example

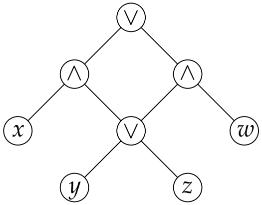

Figure: A DNNF.

Succinctness Relation

Let $\mathrm{S}, \mathrm{T} \subseteq \mathrm{NNF}$.

Say that S is *(polysize) compilable* into T (or T is *at least as succinct as* S) if there exists a polynomial $p\colon \mathbb{N} \to \mathbb{N}$ such that for all $C \in \mathrm{S}$ there exists $D \in \mathrm{T}$ equivalent to C such that

$$\mathrm{size}(D) \leq p(\mathrm{size}(C)).$$

Write $\mathrm{S} \rightsquigarrow \mathrm{T}$ if S is compilable into T, and $\mathrm{S} \not\rightsquigarrow \mathrm{T}$ otherwise.

Succinctness Relation

The succinctness relation is presented in Darwiche and Marquis (2002).

It follows from previous results including

- Quine (1959),
- Chandra and Markowsky (1978),
- Bryant (1986),
- Wegener (1987),
- Gergov and Meinel (1994),
- Gogic, Kautz, Papdimitriou, and Selman (1995),
- Selman and Kautz (1996),
- Cadoli and Donini (1997), and
- Darwiche (1999).

Succinctness Relation

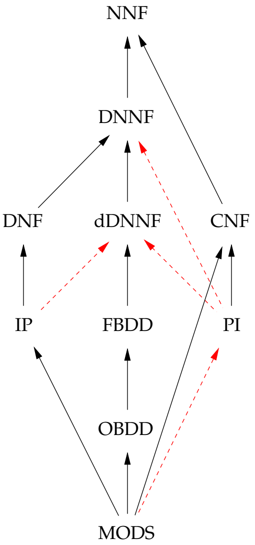

Figure: S ⇢ T means S $\rightsquigarrow$ T unknown.

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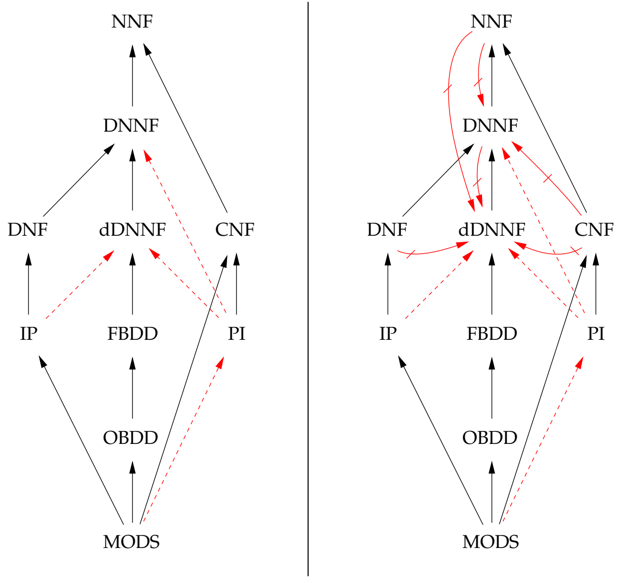

Figure: S $\dashrightarrow$ T means S $\rightsquigarrow$ T unknown. S $\not\rightarrow$ T means S $\not\rightsquigarrow$ T unless PH collapses.

DNNF *vs* CNF

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DNNF $\not\rightsquigarrow$ CNF: $x_1 \oplus \cdots \oplus x_n$ has linear OBDD (and thus DNNF) size, but at least 2^n clauses in any CNF representation (Bryant).

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CNF $\not\rightsquigarrow$ DNNF: If CNF $\rightsquigarrow$ DNNF,
then "clause entailment admits polysize compilation",
then PH collapses (Selman and Kautz; Cadoli and Donini).

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Let $\text{CLIQUE}_n(\mathbf{x})$ be the monotone Boolean function sending its $\binom{n}{2}$ inputs to 1 iff the corresponding n-vertex graph contains a clique on $k(n) = n^{\Omega(1)}$ vertices.

The monotone circuit complexity of CLIQUE_n is weakly exponential in n (Alon and Boppana, 1987).

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Let T be NTM deciding the clique problem in polytime.

Given T, construct for all $n \geq 1$ a CNF $F_n(\mathbf{x}, \mathbf{y})$ of size polynomial in n such that $\exists \mathbf{y} F_n(\mathbf{x}, \mathbf{y})$ computes $\mathrm{CLIQUE}_n(\mathbf{x})$.

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There exists a monotone DNNF computing $\exists \mathbf{y} D_n(\mathbf{x}, \mathbf{y}) \equiv \mathrm{CLIQUE}_n(\mathbf{x})$ having size polynomial in D_n (Darwiche, 2001; Krieger, 2007).

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Hence $D_n(\mathbf{x}, \mathbf{y})$ has size weakly exponential in n.

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CNF $\not\to$ DNNF | *Strongly Exponential*, $2^{\Omega(n)}$

Theorem (B, Capelli, Mengel, Slivovsky)

There exist $c > 0$ and a class $\mathcal{F}$ of CNFs of increasing size such that for all $F \in \mathcal{F}$ and all $D \in$ DNNF equivalent to F,

$$\mathrm{size}(D) \geq 2^{c \cdot \mathrm{size}(F)}.$$

Circuit Complexity

Consequences in circuit complexity.

FBDD: Improve weakly exponential lower bounds on CNF to FBDD compilation (Bollig and Wegener, 1998; Beame et al., 2014).

Multilinear Boolean Circuits: Improve weakly exponential lower bounds (Krieger, 2007).

Knowledge Compilation

Consequences in knowledge compilation.

Corollary

S $\not\rightarrow$ T *for all* $(\mathrm{S}, \mathrm{T}) \in \{\mathrm{PI}, \mathrm{CNF}, \mathrm{NNF}\} \times \{\mathrm{dDNNF}, \mathrm{DNNF}\}$.

Proof.

$\mathcal{F} \subseteq \mathrm{PI}$. The statement follows as
$\mathrm{PI} \subseteq \mathrm{CNF} \subseteq \mathrm{NNF}$ and $\mathrm{dDNNF} \subseteq \mathrm{DNNF}$. □

PI $\not\rightsquigarrow$ DNNF

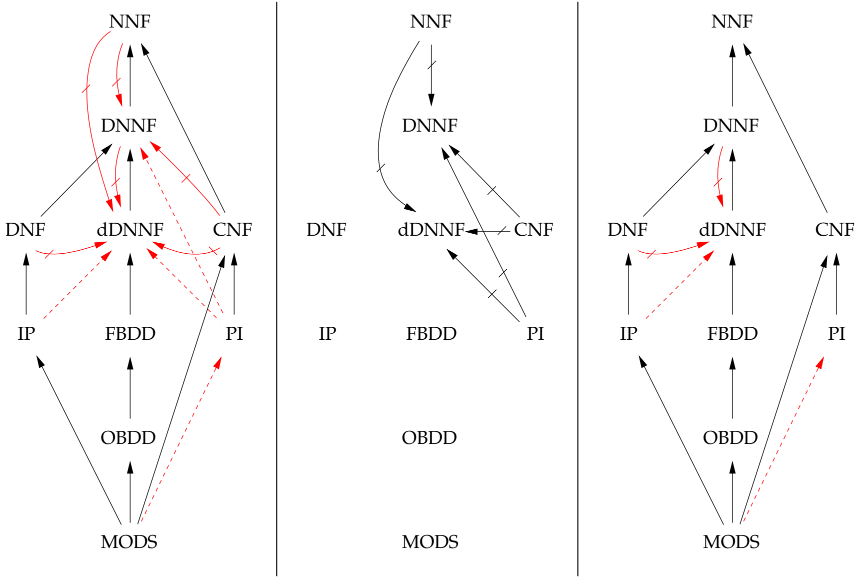

Figure: Status (left), our contribution (center), status modulo our contribution (right).

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A *graph* CNF is a CNF of the form

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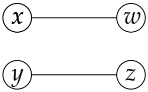

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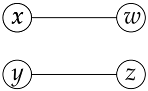

$$G = (\{x, y, w, z\}, \{xw, yz\})$$

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Then:

- $\text{cnf}(G)$ is a monotone Boolean function.
- $\text{cnf}(G)$ is nontrivial, if $|E| \geq 1$.

Nice DNNFs

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Lemma (Krieger)

Let D be a DNNF computing a nontrivial monotone Boolean function f.
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The size of a graph CNF on nice DNNFs is linear in its DNNF size.

Nice DNNFs

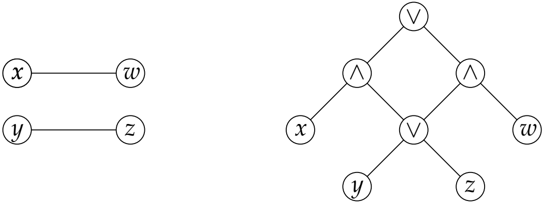

Figure: A nice DNNF (right) computing the vertex covers of a graph (left).

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Theorem (Razgon; B, Capelli, Mengel, Slivovsky)

Let $G = (V, E)$ be a degree d graph and let $I \subseteq V$. Then

$$|\mathrm{vc}(G, I)| \leq 2^{-f(d)|I|} |\mathrm{vc}(G)|$$

where $f(d) = \log_2(1 + 2^{-d}) > 0$.

If $|I|$ is large (linear in $|V|$),
then $\mathrm{vc}(G, I)$ is very small (exponentially small in $|V|$).

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Corollary

Let $G = (V, E)$ be a degree d graph and let $\mathcal{I}$ cover $\mathrm{vc}(G)$. Then

$$|\mathcal{I}| \geq 2^{f(d) \cdot \min\{|I| \,:\, I \in \mathcal{I}\}}$$

where $f(d) = \log_2(1 + 2^{-d}) > 0$.

If $\mathcal{I}$ contains only large sets, then $\mathcal{I}$ is very large.

Proof Strategy

Choose a d-bounded degree graph class $\mathcal{G}$ and $c > 0$ such that, for every $G = (V, E) \in \mathcal{G}$ and every nice DNNF D computing $\mathrm{vc}(G)$ we can find:

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Conclude by the theory of vertex covers that

$$|\mathrm{gates}(D)| \geq S \geq |\mathcal{I}| \geq 2^{f(d) \cdot c|V|}$$

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for $f(d) > 0$ as in the corollary.

Certificates

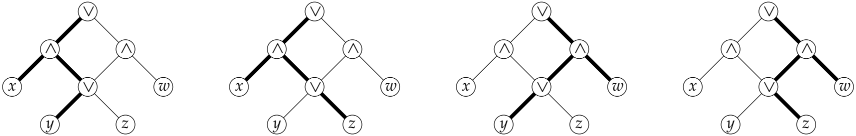

Figure: Certificates for the DNNF displayed in previous examples.

A *certificate* for a DNNF D is a DNNF T defined inductively on D as follows:

- $\text{output}(T) = \text{output}(D)$.
- Let v be a $\wedge$-gate of D with wires from gates v_1 and v_2.
  If v is in T, then both v_1 and v_2 (and their wires to v) are in T.
- Let v be a $\vee$-gate of D with wires from gates v_1 and v_2.
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Then

$$D \equiv \bigvee_{T \in \mathrm{cert}(D)} T$$

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Let $D^{v=0}$ be obtained by relabelling v by 0 in D (and propagating).

$$
\begin{aligned}
D^{v=0} &\equiv \Big(\bigvee_{T \in \mathrm{cert}(T)} T\Big)^{v=0} \\
&\equiv \bigvee_{\{T \in \mathrm{cert}(D)\,:\, v \notin T\}} T \vee \bigvee_{\{T \in \mathrm{cert}(D)\,:\, v \in T\}} T^{v=0} \\
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Call

$$A_{D,v} = \{z \colon z \in \mathrm{vars}(T) \text{ for all } T \in \mathrm{cert}(D) \text{ such that } v \in T\} \subseteq V$$

the set of vertices *agreed* at v in D.

Example

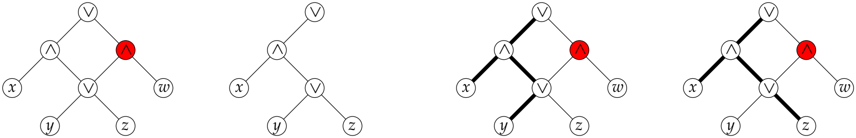

Figure: Eliminating gate $\bullet$ in D gives $D^{\bullet=0}$.
By inspection $\text{cert}(D^{\bullet=0}) = \text{cert}(D) \setminus \{T \in \text{cert}(D) : \bullet \in T\}$.
$A_{D,\bullet} = \{w\}$.

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For the lower bound, we want $|I_i|$ linear in $|V|$.

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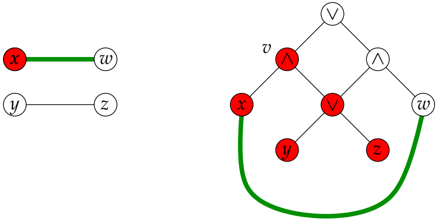

Figure: Graph G (left) has edge xw “across” gate v in its DNNF D (right).

DNNFs and Matchings

$M = \{x_1y_1, \ldots, x_ny_n\}$ matching in G "across" gate v in D.

Claim

For all $i = 1, \ldots, n$, at least one of the following two statements holds:

(1) $x_i \in \mathrm{vars}(T)$ *for all* $T \in \mathrm{cert}(D)$ *such that* $v \in T$.

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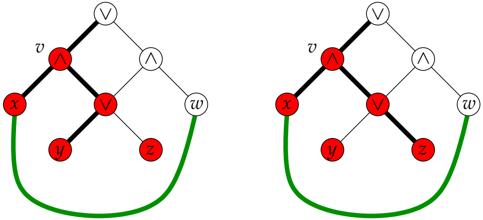

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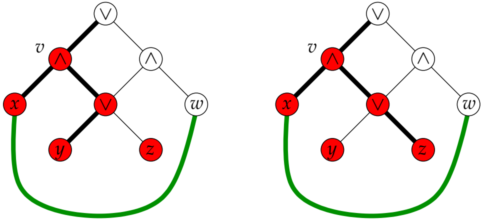

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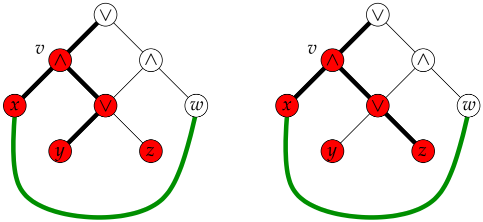

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$|I_M| \geq |M|$ by the claim.

Expander Graphs

A graph $G = (V, E)$ is an (e, d)-expander ($0 < e, d \geq 3$) if:

- G has degree d.
- For all $I \subseteq V$ st $|I| \leq |V|/2$,

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Theorem (Pinsker, 1973)

For every $d \geq 3$ there exist $e > 0$ and a family $\{G_i\}_{i \in \mathbb{N}}$ of graphs of increasing size such that each G_i is an (e, d)-expander.

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Hence D as well.

Thank you for your attention!