A Strongly Exponential Separation of DNNFs from CNFs

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joint work with
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Outline

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**Motivation**

In choosing a *representation language* for a propositional theory there is a trade-off between “succinctness” and “tractability”.

Darwiche and Marquis (2002) systematically investigate a hierarchy of representation languages that strike this balance in different ways.

**Contribution**

**Proof**
**Representation Languages**

*Figure:* Inclusion relation on representation languages (Hasse diagram).
**Representation Languages**

**Negation Normal Forms (NNF)**  Boolean circuits having unbounded fanin AND and OR gates with negations pushed to the input gates.

**Decomposable NNFs (DNNF)**  NNFs where subcircuits leading into each AND gate are defined on disjoint sets of variables.

**Deterministic DNNFs (dDNNF)**  DNNFs where subcircuits leading into each OR gate never simultaneously evaluate to 1.

**Conjunctive Normal Forms (CNF)**  NNFs where...

**Prime Implicate Forms (PI)**  CNFs where entailed clauses are already entailed by a single clause in the CNF and no clause in the CNF is entailed by another.

... ... ...

size(C) is the number of arcs in the DAG underlying C (for C in NNF).
Example

Figure: A DNNF.
Let $S, T \subseteq \text{NNF}$.

Say that $S$ is \textit{(polysize) compilable} into $T$ (or $T$ is \textit{at least as succinct as} $S$) if there exists a polynomial $p : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $C \in S$ there exists $D \in T$ equivalent to $C$ such that

$$\text{size}(D) \leq p(\text{size}(C)).$$

Write $S \leadsto T$ if $S$ is compilable into $T$, and $S \not\leadsto T$ otherwise.
The succinctness relation is presented in Darwiche and Marquis (2002).

It follows from previous results including

- Quine (1959),
- Chandra and Markowsky (1978),
- Bryant (1986),
- Wegener (1987),
- Gergov and Meinel (1994),
- Gogic, Kautz, Papdimitriou, and Selman (1995),
- Selman and Kautz (1996),
- Cadoli and Donini (1997), and
- Darwiche (1999).
**Succinctness Relation**

![Diagram showing the succinctness relation among different classes of Boolean functions: MODS, OBDD, DNNF, DNF, dDNNF, NNF, IP, FBDD, PI, OBDD, and MODS.](image)

*Figure:* $S \rightarrow T$ means $S \sim T$ unknown.
**Motivation**

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**Succinctness Relation**

[Diagram showing the succinctness relation between different logical forms: NNF, DNNF, OBDD, dDNNF, IP, FBDD, PI, MODS, DNF, CNF.]

*Figure:* S $\rightarrow\rightarrow$ T means S $\sim\sim$ T unknown. S $\not\rightarrow\rightarrow$ T means S $\not\sim\sim$ T unless PH collapses.
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**DNNF vs CNF**

- $DNNF \not\Rightarrow CNF$: $x_1 \oplus \cdots \oplus x_n$ has linear OBDD (and thus DNNF) size, but at least $2^n$ clauses in any CNF representation (Bryant).

- $CNF \not\Rightarrow DNNF$: If CNF $\Rightarrow DNNF$, then "clause entailment admits polysize compilation", then PH collapses (Selman and Kautz; Cadoli and Donini).
DNNF vs CNF

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<td>$\text{CNF} \not\sim \text{DNNF}$</td>
<td>Weakly Exponential, $2^{n\Omega(1)}$</td>
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Let CLIQUE\(_n(x)\) be the monotone Boolean function sending its \(\binom{n}{2}\) inputs to 1 iff the corresponding \(n\)-vertex graph contains a clique on \(k(n) = n^{\Omega(1)}\) vertices.

The monotone circuit complexity of CLIQUE\(_n\) is weakly exponential in \(n\) (Alon and Boppana, 1987).
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Let \(T\) be NTM deciding the clique problem in polytime.

Given \(T\), construct for all \(n \geq 1\) a CNF \(F_n(x, y)\) of size polynomial in \(n\) such that \(\exists yF_n(x, y)\) computes CLIQUE\(_n\)\((x)\).
Let $\text{CLIQUE}_n(x)$ be the monotone Boolean function sending its $\binom{n}{2}$ inputs to 1 iff the corresponding $n$-vertex graph contains a clique on $k(n) = n^{\Omega(1)}$ vertices.

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Let $D_n(x, y)$ be a DNNF computing $F_n(x, y)$. 
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There exists a monotone DNNF computing $\exists y D_n(x, y) \equiv \text{CLIQUE}_n(x)$ having size polynomial in $D_n$ (Darwiche, 2001; Krieger, 2007).
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Hence $D_n(x, y)$ has size weakly exponential in $n$. 
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<td>CNF ↳ DNNF</td>
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Theorem (B, Capelli, Mengel, Slivovsky)

There exist $c > 0$ and a class $\mathcal{F}$ of CNFs of increasing size such that for all $F \in \mathcal{F}$ and all $D \in \text{DNNF}$ equivalent to $F$,

$$\text{size}(D) \geq 2^{c \cdot \text{size}(F)}.$$
Consequences in circuit complexity.

**FBDD**  Improve weakly exponential lower bounds on CNF to FBDD compilation (Bollig and Wegener, 1998; Beame et al., 2014).

**Multilinear Boolean Circuits**  Improve weakly exponential lower bounds (Krieger, 2007).
Consequences in knowledge compilation.

**Corollary**

\[ S \not\rightarrow T \text{ for all } (S, T) \in \{\text{PI, CNF, NNF}\} \times \{\text{dDNNF, DNNF}\}. \]

**Proof.**

\( \mathcal{F} \subseteq \text{PI}. \) The statement follows as \( \text{PI} \subseteq \text{CNF} \subseteq \text{NNF} \) and \( \text{dDNNF} \subseteq \text{DNNF}. \)
**PI \not\leftrightarrow DNNF**

*Figure:* Status (left), our contribution (center), status modulo our contribution (right).
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- Proof
A graph CNF is a CNF of the form

\[ \text{cnf}(G) = \bigwedge_{xy \in E} x \lor y \]

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\[ \begin{array}{c}
\text{x} \quad \text{w} \\
\text{y} \quad \text{z}
\end{array} \]

\[ G = (\{x, y, w, z\}, \{xw, yz\}) \]
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$G = (\{x, y, w, z\}, \{xw, yz\})$

$$\text{cnf}(G) = (x \lor w) \land (y \lor z)$$
Graph CNFs

Let \( \text{vc}(G) \) denote the vertex covers of graph \( G \).
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Let $vc(G)$ denote the vertex covers of graph $G$.

Then

$$\text{mod}(\text{cnf}(G)) = vc(G)$$
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Then

\[
\text{mod(cnf}(G)) = vc(G)
\]

Then:

- \( cnf(G) \) is a monotone Boolean function.
- \( cnf(G) \) is nontrivial, if \( |E| \geq 1 \).
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**Lemma (Krieger)**

Let $D$ be a DNNF computing a nontrivial monotone Boolean function $f$. There exists a nice DNNF $D'$ equivalent to $D$ st

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**Proof (Sketch).**

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The size of a graph CNF on nice DNNFs is linear in its DNNF size.
**Nice DNNFs**

*Figure:* A nice DNNF (right) computing the vertex covers of a graph (left).
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**Vertex Covers**

Which fraction of vertex covers of a graph contain a fixed subset of vertices?

For $G = (V, E)$ and $I \subseteq V$, write $vc(G,I) = \{C \in vc(G) : I \subseteq C\}$ for the vertex covers of $G$ containing $I$.

**Theorem (Razgon; B, Capelli, Mengel, Slivovsky)**

Let $G = (V, E)$ be a degree $d$ graph and let $I \subseteq V$. Then $|vc(G,I)| \leq 2 - f(d)|I| |vc(G)|$ where $f(d) = \log_2(1 + 2 - d) > 0$.

If $|I|$ is large (linear in $|V|$), then $vc(G,I)$ is very small (exponentially small in $|V|$).
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**Theorem (Razgon; B, Capelli, Mengel, Slivovsky)**

Let $G = (V, E)$ be a degree $d$ graph and let $I \subseteq V$. Then

$$|vc(G, I)| \leq 2^{-f(d)|I|} |vc(G)|$$

where $f(d) = \log_2(1 + 2^{-d}) > 0$.

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A family $\mathcal{I}$ of subsets of $V$ covers $\text{vc}(G)$ if $\text{vc}(G) = \bigcup_{I \in \mathcal{I}} \text{vc}(G, I)$. 
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**Corollary**

Let $G = (V, E)$ be a degree $d$ graph and let $\mathcal{I}$ cover $vc(G)$. Then

$$|\mathcal{I}| \geq 2^{f(d) \cdot \min\{|I| : I \in \mathcal{I}\}}$$

where $f(d) = \log_2(1 + 2^{-d}) > 0$.

If $\mathcal{I}$ contains only large sets, then $\mathcal{I}$ is very large.
Proof Strategy

Choose a \( d \)-bounded degree graph class \( \mathcal{G} \) and \( c > 0 \) such that, for every \( G = (V, E) \in \mathcal{G} \) and every nice DNNF \( D \) computing \( \text{vc}(G) \) we can find:
Choose a $d$-bounded degree graph class $\mathcal{G}$ and $c > 0$ such that, for every $G = (V, E) \in \mathcal{G}$ and every nice DNNF $D$ computing $vc(G)$ we can find:

- $S$ distinct gates in $D$
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Conclude by the theory of vertex covers that

$$|\text{gates}(D)| \geq S \geq |\mathcal{I}| \geq 2^{f(d) \cdot c|V|}.$$
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Conclude by the theory of vertex covers that

$$|\text{gates}(D)| \geq S \geq |\mathcal{I}| \geq 2^{f(d) \cdot c|V|}$$

for $f(d) > 0$ as in the corollary.
A certificate for a DNNF $D$ is a DNNF $T$ defined inductively on $D$ as follows:

- output($T$) = output($D$).
- Let $v$ be a $\land$-gate of $D$ with wires from gates $v_1$ and $v_2$. If $v$ is in $T$, then both $v_1$ and $v_2$ (and their wires to $v$) are in $T$.
- Let $v$ be a $\lor$-gate of $D$ with wires from gates $v_1$ and $v_2$. If $v$ is in $T$, then exactly one of $v_1$ and $v_2$ (and its wire to $v$) is in $T$.
Certificates

Figure: Certificates for the DNNF displayed in previous examples.

A certificate for a DNNF $D$ is a DNNF $T$ defined inductively on $D$ as follows:

- **output**($T$) = **output**($D$).
- Let $v$ be a $\wedge$-gate of $D$ with wires from gates $v_1$ and $v_2$.
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- Let $v$ be a $\lor$-gate of $D$ with wires from gates $v_1$ and $v_2$.
  If $v$ is in $T$, then exactly one of $v_1$ and $v_2$ (and its wire to $v$) is in $T$. 
For a DNNF $D$, write $\text{cert}(D) = \{T : T \text{ certificate of } D\}$. 
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Then

$$D \equiv \bigvee_{T \in \text{cert}(D)} T$$
Gate Elimination

Let $D$ be a DNNF such that $\text{mod}(D) \subseteq \text{vc}(G)$ and $v$ be a gate in $D$. 
Gate Elimination

Let $D$ be a DNNF st $\text{mod}(D) \subseteq \text{vc}(G)$ and $v$ be a gate in $D$.

Let $D^{v=0}$ be obtained by relabelling $v$ by 0 in $D$ (and propagating).

\[
D^{v=0} \equiv ( \bigvee_{T \in \text{cert}(T)} T )^{v=0} \\
\equiv \bigvee \{ T \in \text{cert}(D) : v \notin T \} \lor \bigvee \{ T \in \text{cert}(D) : v \in T \} \\
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\[
\equiv \bigvee \{T \in \text{cert}(D) : v \not\in T\}
\]

Call

\[
A_{D,v} = \{z : z \in \text{vars}(T) \text{ for all } T \in \text{cert}(D) \text{ such that } v \in T\} \subseteq V
\]

the set of vertices agreed at $v$ in $D$. 
Example

Figure: Eliminating gate \( \bullet \) in \( D \) gives \( D^{\bullet} = 0 \).

By inspection \( \text{cert}(D^{\bullet}=0) = \text{cert}(D) \setminus \{T \in \text{cert}(D) : \bullet \in T\} \).

\( A_{D,\bullet} = \{w\} \).
Gate Elimination

Let $G = (V, E)$ be a graph, and $D$ be a nice DNNF computing $\text{vc}(G)$. 
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Let $v_1\ldots,v_S$ be distinct gates in $D$, and $D_0, D_1, D_2, \ldots, D_S$ be DNNFs such that:
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Let $I_i \subseteq A_{D_{i-1}, v_i}$. 
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Then $\mathcal{I} = \{I_i : i = 1, \ldots, S\}$ covers $\text{vc}(G)$. 
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- $D_s \equiv 0$

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Then $\mathcal{I} = \{I_i: i = 1, \ldots, S\}$ covers $vc(G)$.

For the lower bound, we want $|I_i|$ linear in $|V|$. 
Let $D$ be a nice DNNF such that $\text{mod}(D) \subseteq \text{vc}(G)$. 
DNNFs and Matchings

Let $D$ be a nice DNNF such that $\text{mod}(D) \subseteq \text{vc}(G)$.

Let $v$ be a gate in $D$ and $D_v$ be the subcircuit of $D$ rooted at $v$ (think of $v$ as a candidate for elimination).
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Let $M = \{x_1y_1, \ldots, x_ny_n\}$ be a matching in $G$ with $\{x_1, \ldots, x_n\} \subseteq \text{vars}(D_v)$ and $\{y_1, \ldots, y_n\} \subseteq \text{vars}(D) \setminus \text{vars}(D_v)$. 

![Figure: Graph $G$ (left) has edge $xw$ across gate $v$ in its DNNF $D$ (right).]
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**Figure:** Graph $G$ (left) has edge $xw$ “across” gate $v$ in its DNNF $D$ (right).
DNNFs and Matchings

\[ M = \{x_1y_1, \ldots, x_ny_n\} \text{ matching in } G \text{ “across” gate } v \text{ in } D. \]

**Claim**

For all \( i = 1, \ldots, n \), at least one of the following two statements holds:

1. \( x_i \in \text{vars}(T) \) for all \( T \in \text{cert}(D) \) such that \( v \in T. \)
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\[ I_M = \{x_i: i \in [n] \text{ such that (1) holds}\} \cup \{y_i: i \in [n] \text{ such that (2) holds}\} \subseteq A_{D,v}. \]
Motivation

**Claim**

For all $i = 1, \ldots, n$, at least one of the following two statements holds:

1. $x_i \in \text{vars}(T)$ for all $T \in \text{cert}(D)$ such that $v \in T$.
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$I_M = \{x_i : i \in [n] \text{ such that (1) holds}\} \cup \{y_i : i \in [n] \text{ such that (2) holds}\} \subseteq A_{D,v}$.

$|I_M| \geq |M|$ by the claim.
A graph $G = (V, E)$ is an $(e, d)$-expander ($0 < e, d \geq 3$) if:

- $G$ has degree $d$.
- For all $I \subseteq V$ st $|I| \leq |V|/2$,
  \[ |N_I| \geq e|I| \]

where $N_I$ is the neighbourhood of $I$ in $G$. 
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**Theorem (Pinsker, 1973)**

For every $d \geq 3$ there exist $e > 0$ and a family $\{G_i\}_{i \in \mathbb{N}}$ of graphs of increasing size such that each $G_i$ is an $(e, d)$-expander.
Lemma

Let $G = (V, E)$ be a $(e, d)$-expander and $D$ be a nice DNNF st $\text{mod}(D) \subseteq \text{vc}(G)$. There exists $v \in D$ and $I \subseteq A_{D,v}$ such that $|I|$ is linear in $|V|$.
Expander Graphs

Lemma

Let $G = (V, E)$ be a $(e, d)$-expander and $D$ be a nice DNNF st $\text{mod}(D) \subseteq \text{vc}(G)$. There exists $v \in D$ and $I \subseteq A_{D,v}$ such that $|I|$ is linear in $|V|$.

Proof (Idea).

$|C| \geq |V| / (d+1)$ for all $C \in \text{vc}(G)$.

Find (greedily) $v \in D$ st $|V| / (d+1) \leq |\text{vars}(D_v)| \leq |V| / 2$.

$|N_{\text{vars}}(D_v)| \geq e |\text{vars}(D_v)| = \Omega(|V|)$.

Find matching $M$ in $G$ of size $\Omega(|V|)$ between $\text{vars}(D_v)$ and $N_{\text{vars}}(D_v) \subseteq \text{vars}(D) \setminus \text{vars}(D_v)$.

$I = I_M \subseteq A_{D,v}$.

$|I| \geq |M| = \Omega(|V|)$.
## Expander Graphs

### Lemma

Let $G = (V, E)$ be a $(e, d)$-expander and $D$ be a nice DNNF st $\text{mod}(D) \subseteq \text{vc}(G)$. There exists $v \in D$ and $I \subseteq A_{D,v}$ such that $|I|$ is linear in $|V|$.

### Proof (Idea).

- $|C| \geq |V|/(d + 1)$ for all $C \in \text{vc}(G)$. 
**Lemma**

Let $G = (V, E)$ be a $(e, d)$-expander and $D$ be a nice DNNF st $\text{mod}(D) \subseteq \text{vc}(G)$. There exists $v \in D$ and $I \subseteq A_{D,v}$ such that $|I|$ is linear in $|V|$.

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**Lemma**

Let $G = (V, E)$ be a $(e, d)$-expander and $D$ be a nice DNNF st mod($D$) $\subseteq$ vc($G$). There exists $v \in D$ and $I \subseteq A_{D,v}$ such that $|I|$ is linear in $|V|$.

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**Expander Graphs**
## Expander Graphs

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Expander Graphs

**Lemma**

Let $G = (V, E)$ be a $(e, d)$-expander and $D$ be a nice DNNF such that $\text{mod}(D) \subseteq \text{vc}(G)$. There exists $v \in D$ and $I \subseteq A_{D,v}$ such that $|I|$ is linear in $|V|$.

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□
Proof Sketch

Let \( G = (V, E) \) be a \((e, d)\)-expander and \( D \) a nice DNNF computing \( vc(G) \).

Find \( v_1 \in D \) and \( I_1 \subseteq \mathcal{A} \) such that \(|I_1|\) is linear in \(|V|\).

Eliminate \( v_1 \) to obtain \( D_1 = D_{v_1} = 0 \).

Iterate, unless \( D_1 \equiv 0 \).

\[ I = \{I_1, I_2, \ldots\} \] covers \( vc(G) \).

\(|I|\) is exponentially large in \(|V|\).

Hence \( D \) as well.
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Let $G = (V, E)$ be a $(e, d)$-expander and $D$ a nice DNNF computing $\text{vc}(G)$. 
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Thank you for your attention!