Detecting and Exploiting Subproblem Tractability

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Background

- Suppose you are all *familiar* with constraint satisfaction?
  - Given: variables each with a (finite) domain of possible values and a set of constraints (relations on allowed values for tuple of vars)
  - Question: does there exist assignment of values to variables so that every constraint is satisfied?

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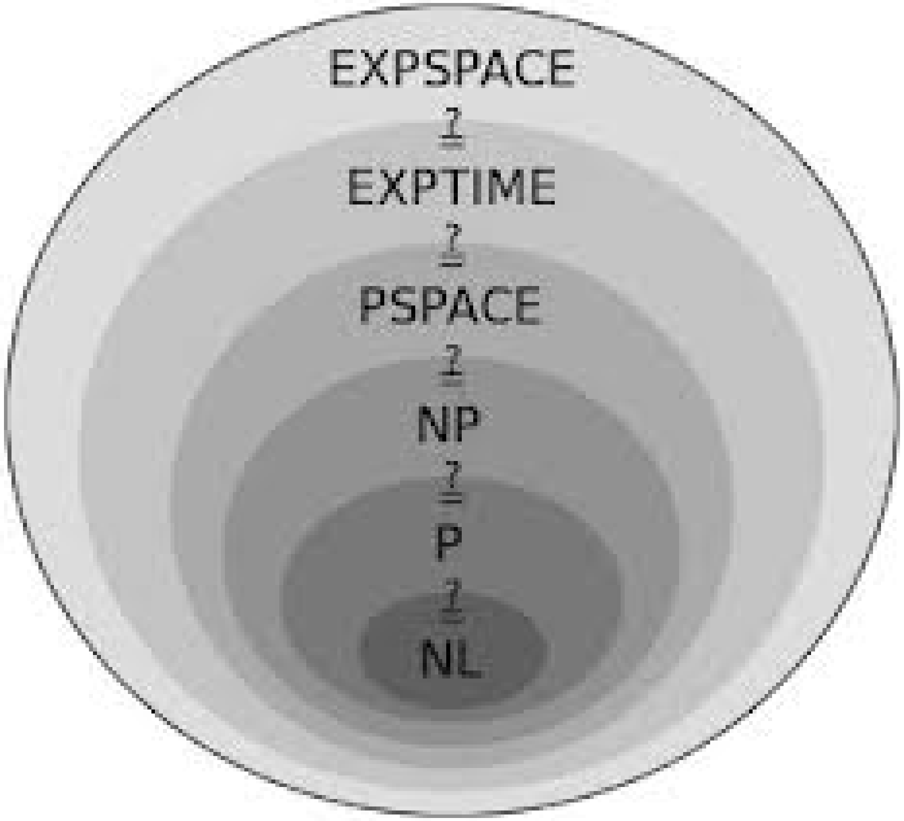
    - Graph colouring: vars=nodes, values=colours, constraints=nodes connected by an edge have different colours
    - SAT: vars=Boolean, values=true/false, constraints=clauses
    - Scheduling: vars=jobs, values=start times, constraints=start times respect release times, start times+job lengths respect due dates, etc.

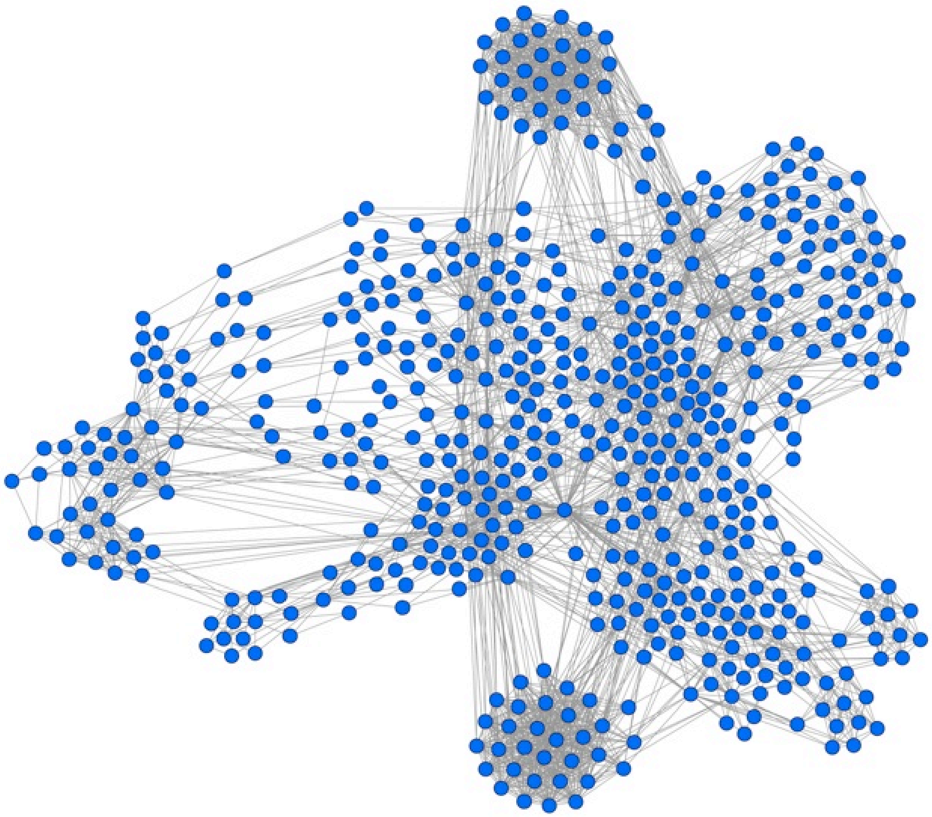
Motivation

- Lots of research on tractable constraint problems
  - Restricted language (e.g. 2SAT)
  - Restricted constraint structure (e.g. tree)
- But solvers often perform poorly on tractable problems
  - [Petke & Jeavons 2009]
  - Little research on detecting when a (sub)problem is tractable



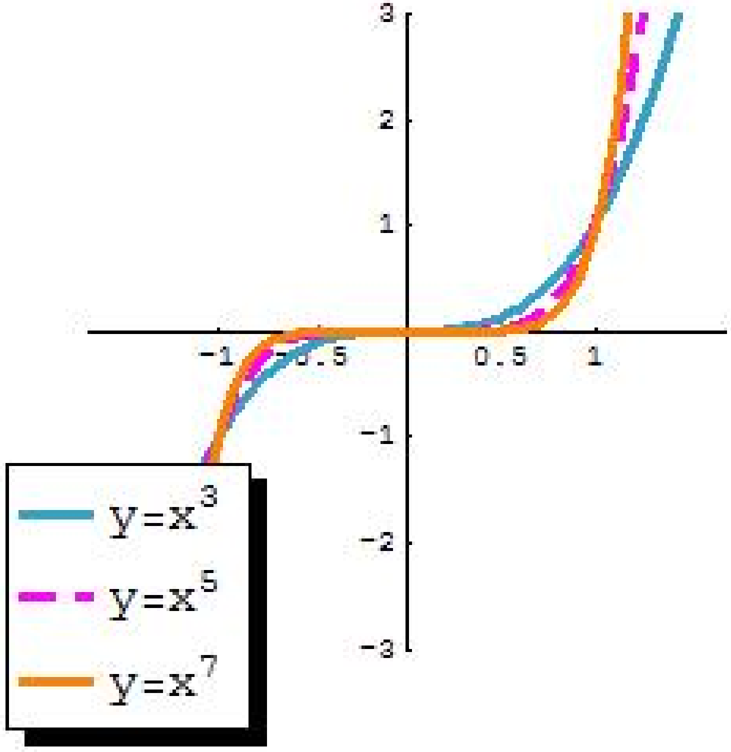
Motivation

- Exploit (strong) backdoors into tractable subproblems
  - Identify some key variables (backdoor) that make problem intractable
  - Branch on these to give a tractable subproblem
  - FPT algorithm in size of backdoor
- Need to detect tractable subproblems
  - Not so much work on computational question of how to identify tractable subproblems!



Motivation

- Preliminary work
  - Our methods for identifying tractable subproblems have large polynomial cost
    - E.g. $O(d^6)$ and $O(d^7)$ time
  - May be able to offset this over many instances
  - Challenge will be to reduce costs!



Outline

- Identify tractable classes
- Exploit tractable classes

+ Outline

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  - Detecting set of relations that admit majority polymorphism
  - Detecting set of relations that admit conservative Mal'tsev polymorphism
- Exploit tractable classes

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  - Detecting set of relations that admit majority polymorphism
  - Detecting set of relations that admit conservative Mal'tsev polymorphism
- Exploit tractable classes
  - FPT algorithm for idempotent classes
  - FPT algorithm for conservative classes
  - NP-hardness when we don't know backdoor and tractable subset of language
    - But FPT in d+k+r

Identifying tractable class

- Constraint problems are tractable if their relations are closed under majority polymorphisms

  [Jeavons et al 1997]

Language closed under majority polymorphism = generalization of 2-SAT and 0/1/all constraints

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  [Bulatov & Dalmau 2006]

Language closed under majority polymorphism = generalization of 2-SAT and 0/1/all constraints

Language closed under Mal'tsev polymorphism = generalization of linear equations over a field

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- Thm: Can decide if language is closed under majority polymorphism in $O(d^7t^4)$ time
  - Proof: Build an indicator problem, repeatedly apply SAC until failure/solution.

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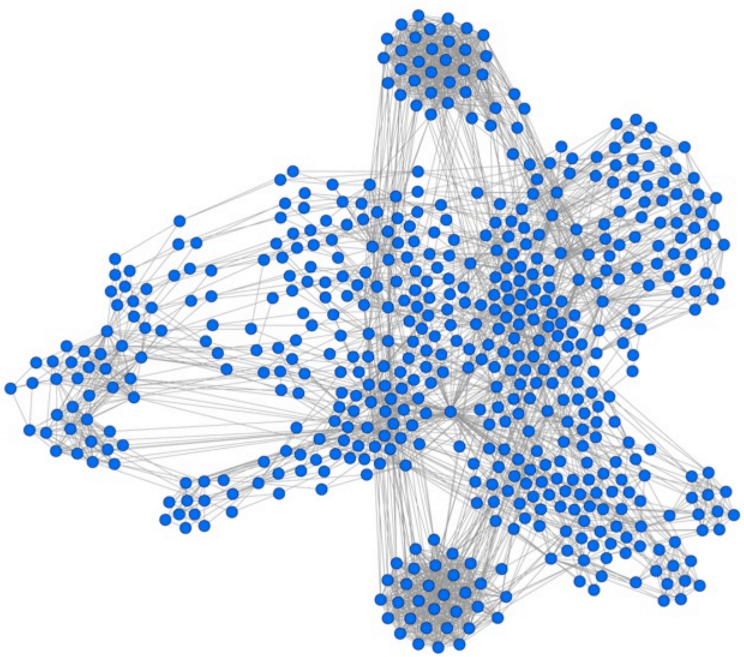
- Thm: Can decide if language is closed under majority polymorphism in $O(d^7t^4)$ time
  - Proof: Build an indicator problem, repeatedly apply SAC until failure/solution.

- Thm: Can decide if language is closed under a conservative Mal'tsev polymorphism in $O(d^6)$ time
  - Proof: Build a special indicator problem, repeatedly enforce AC, merge equals, remove redundant/universal constraints until failure/sol.

Exploiting tractable class

- Inspired by cycle cutset method [Dechter & Pearl 1987]
  - Instantiate variables to cut cycles
  - Then decide backtrack free with Directional AC

- Tractable subproblem based on structure of network
  - We now do much the same with a tractable language



Exploiting tractable class

- Idempotent class
  - I.e. fixing variables, we remain within the class

- Conservative class
  - Closed under *all* unary constraints
  - Stronger condition, smaller FPT algorithm

Exploiting tractable class

- Idempotent class
  - Let constraint relations $C = C_1 + C_2$
    - Where C_2 closed under the idempotent polymorphism
    - Instantiate all m vars in C_1
    - Leaves tractable subproblem made from C_2 and instantiations
    - FPT in d+m
- Conservative class

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- Conservative class
  - Similar algorithm
  - FPT in strictly smaller parameter, d+k
    - Where k is minimum vertex cover of primal graph of C_1

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- What if we need to search simultaneously for a backdoor and the tractable relations?

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  - Thm: NP-hard to decide if C partitions into C_1+C_2 such that C_2 admits a conservative majority polymorphism and C_1 has a vertex cover of at most k
    - In fact, W[2]-hard in k

Exploiting tractable class

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- What if we need to search simultaneously for a backdoor and the tractable relations?
  - Thm: FPT in d+k+r to decide if C partitions into C_1+C_2 such that C_2 admits a conservative majority polymorphism and C_1 has a vertex cover of at most k

Empirical results

- Tested instances of 4th Constraint Solver Competition
  - Limited to those without globals
  - All instances put in extensional form
  - 191 series of instances
  - Tested for existence of subproblem closed under conservative majority polymorphism

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- Tested instances of 4th Constraint Solver Competition
  - Limited to those without globals
  - All instances put in extensional form
  - 191 series of instances
  - Tested for existence of subproblem closed under conservative majority polymorphism
- Results
  - 135 series: exhausted 8Gb of memory
  - 40 series: large backdoor
  - But a few promising series
    - E.g. 5 *prime* series
    - ¼ had small backdoor (0 to 6 vars out of 100)

Conclusions

- We can exploit constraint problems that are *nearly* tractable
- Branch on backdoor into a tractable language
- For such methods to be useful, we need methods to identify tractable (sub)languages
  - Propose here two polynomial methods to identify language closed under a majority polymorphism, and under a conservative Mal'tsev polymorphism
- Computing a backdoor into a language closed under a conservative majority polymorphism is W[2]-hard in k, but FPT in d+k+r

Questions?

- PS we’re recruiting PhD students and a PostDoc
- Good student = guaranteed funding
- Shortly after graduating -> Australian citizen