The structure of impossibility results in abstract aggregation theory

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- Determinants: properties of the mapping and properties of the involved spaces.

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- Aggregation: Mapping $f : X^n \longrightarrow Y$ from a product space of n individual factor spaces (individual preferences) into a lower dimensional outcome space (social preference or alternatives)
- Determinants: properties of the mapping and properties of the involved spaces.
- Highly relevant in many areas, but abundant in negative results: impossibility of satisfying sets of mild looking and highly desirable properties.

Two classical results in social choice theory

- Arrow's (1951) general (im)possibility theorem: The only rule for aggregating the preferences of a set of more than two individuals into a collective preference (social welfare function) which satisfies universal domain, the Pareto property, and **independence** of irrelevant alternatives is the dictatorship of a particular individual.

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- Gibbard/Satterthwaite (1973/1975): The only rule for aggregating the preferences of a set of more than two individuals into a collectively chosen alternative (social choice function) which is **strategy-proof** and onto is the dictatorship of a particular individual.

Extensions and applications of classical social choice theory

- Generalization of the problem of aggregation from preferences to arbitrary information (for a survey see List and Puppe 2009)

-Recently (2002-): Judgment aggregation (individual characteristics = judgment (belief) sets
(Dietrich and List, Nehring and Puppe, Dokow and Holzman)
- Before, but isolated: abstract aggregation: Guilbaud 1952, Wilson 1975, Rubinstein and Fishburn 1986

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- Logical and algorithmic analysis of aggregation problems (computational social choice, for a survey see Rothe et al. 2011): Construction and analysis of formal languages for aggregation problems (e.g. Pauly, Endriss), computational complexity of aggregation rules (Hemaaspandra and Hemaaspandra)

Impossibility in judgment aggregation

- the typical example: the discursive dilemma

	p	q	$p \wedge q$
1	1	1	1
2	1	0	0
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- and the typical impossibility result in judgment aggregation:
  For a sufficiently rich agenda of logically interconnected propositions the only aggregation rule which satisfies conditions analog to Arrow's theorem is a dictatorship.

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Metatheorems for impossibility theorems

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- operate by formulating the problem of strategyproof social choice functions in the framework of the aggregation of preferences (social welfare functions): Gibbard-Satterthwaite type results obtained in Arrow type framework
- But: for generalizations of aggregation problems, it might be the other way round: Arrow type results better obtained from Gibbard-Satterthwaite type approach

Intuition and significance of the metatheorem

- Two desirable properties
  - **Non-imposition**: every possible social outcome must be obtainable at some profile of individual opinions
  - **Individual responsiveness**: Every change of the social outcome must be induced by a corresponding change in the opinion of an individual (pivotality)

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  - **Individual responsiveness**: Every change of the social outcome must be induced by a corresponding change in the opinion of an individual (pivotality)
- **Theorem**: For an abstract aggregation problem with an agenda of sufficient logical richness the only non-imposed aggregation rule which satisfies individual responsiveness is the dictatorship of a particular individual.

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- Observe: for every $X \subseteq \{0,1\}^P$ there exists a set P of propositions such that X is the set of logically consistent valuations of P. (E.g. the 16 elements of the power set of $\{0,1\}^2$ reprensenting all logical connections between two propositions)

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- Thus $\{0,1\}^P \backslash X$ is the set of **inadmissible** valuations
- X^N is the set of all **profiles** of admissible individual valuations ${}^i x = ({}^i x_1, ..., {}^i x_j, ..., {}^i x_{|P|})$

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- For convenience set of propositions identified with its index set, the set of **issues**
- Thus $\{0,1\}^P \backslash X$ is the set of **inadmissible** valuations
- X^N is the set of all **profiles** of admissible individual valuations ${}^i x = ({}^i x_1, \ldots, {}^i x_j, \ldots, {}^i x_{|P|})$
- **Aggregation rule** $f : X^N \rightarrow X$

Properties of aggregation rules (some notation and terminology)

- For any profile $x \in X^N$, any issue $j \in P$ and any value $v \in \{0, 1\}$, the set
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collects the individuals that assign the value v to the issue j.
- For any judgment aggregation rule $f : X^N \to X$, f_j denotes the j-th component of f, i.e. the function
$$f_j : X^N \to \{0, 1\}$$
that assigns to any profile of individual valuations the social valuation of the issue j.

Properties of aggregation rules

- An aggregation rule $f : X^N \to X$ is **nonimposed** if $f(X^N) = X$
- An aggregation rule $X^N \to X$ satisfies **individual responsiveness** if for any profiles $x, x' \in X^N$, any issue $j \in P$ and any value $v \in \{0,1\}$ $f_j(x) = v \wedge f_j(x') = 1 - v$ implies that there exists a pivotal individual $i \in N$ such that ${}^i x_j = v \wedge^i x_j' = 1 - v$
- An aggregation rule $X^N \to X$ is monotonic if for any profiles $x, x' \in X^N$, any issue $j \in P$ and any value $v \in \{0,1\}$ $f_j(x) = v \wedge x_j(v) \subset x_j'(v)$ implies $f_j(x) = v$.
- *An aggregation rule* $f : X^N \to X$ *is* **independent** *if for any issue* $j \in P$, *any valuation* $v \in \{0,1\}$, *and for all profiles* $x, x^{'} \in X^N$

$$f_j(x) = v \Rightarrow \left[x_j(v) = x_j^{'}(v) \Rightarrow f_j(x^{'}) = v \right].$$

Winning coalitions

- Given an aggregation rule $f : X^N \to X$, a set of individuals $U = x_j(v) \in 2^N$ is a **winning coalition** for an issue $j \in P$, a valuation $v \in \{0, 1\}$ and a profile $x \in X^N$ if $f_j(x) = v$ (i.e. if its valuation determines the collective valuation).

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- When f is independent the set U is also a winning coalition for j, v and any profile $x^{'} \in X^N$ such that $x_j^{'}(v) = x_j(v)$. Thus, an independent aggregation rule can be characterised by identifying for any issue $j \in P$ and any value $v \in \{0, 1\}$ the induced family $\mathcal{W}_j^v = \{U \in 2^N : x_j(v) = U \Rightarrow f_j(x) = v\}$, i.e. the family of all coalitions that are winning for a given issue and a given valuation.

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- When f is independent the set U is also a winning coalition for j, v and any profile $x' \in X^N$ such that $x'_j(v) = x_j(v)$. Thus, an independent aggregation rule can be characterised by identifying for any issue $j \in P$ and any value $v \in \{0, 1\}$ the induced family $\mathcal{W}_j^v = \{U \in 2^N : x_j(v) = U \Rightarrow f_j(x) = v\}$, i.e. the family of all coalitions that are winning for a given issue and a given valuation.
- *An aggregation rule $f : X^N \rightarrow X$ is* **paretian** *if for any issue $j \in P$, any valuation $v \in \{0, 1\}$, $N \in \mathcal{W}_j^v$.*

Towards a proof

Lemma 1: *A non-imposed aggregation rule $f : X^N \to X$ which satisfies individual responsiveness is independent.*

Proof: For a proof by contraposition, assume to the contrary that there exist $x, x' \in X$, $j \in P$, $v \in \{0, 1\}$ s.t. $f_j(x) = 1 - v$, albeit $f_j(x') = v$ and $x_j(v) = x'_j(v)$.

The latter implies that there does not exist a pivotal $i \in N$ s.t. ${}^i x_j = 1 - v$ and ${}^i x'_j = v$, contradicting individual responsiveness.

Towards a proof (ctd)

Lemma 2: *A non-imposed aggregation rule $f : X^N \rightarrow X$ which satisfies individual responsiveness (and is hence independent) is also monotonic.*
Proof: For a proof by contraposition assume that there exist i-variants $x, x' \in X$, $j \in P$, $v \in \{0, 1\}$ such that ${}^i x_j = 1 - v$ and ${}^i x'_j = v$ but $f_j(x) = v$ and $f_j(x') = 1 - v$, which contradicts individual responsiveness.

Proof ctd

Lemma 3: *An independent and monotonic aggregation rule* $f : X^N \rightarrow X$ *satisfies individual responsiveness.*

Proof: For a proof by contraposition assume to the contrary that for some $j \in P$, $v \in \{0, 1\}, x \in X$ with $f_j(x) = v$ there exists $x' \in X$ s.t. $f_j(x') = 1 - v$, but either $x_j(v) = x'_j(v)$, violating independence, or $(\mathcal{W}_j^v \ni)x_j(v) \subset x'_j(v)$, violating monotonicity.

Observation: In the presence of nonimposition, individual responsiveness is equivalent to the conjunction of monotonicity and independence.

Proof metatheorem last step

- **Observation** (canonical theorem of judgment aggregation): For a sufficiently rich ("totally blocked") agenda the only independent and monotonic aggregation rule is the dictatorship of a particular individual.

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- **Observation** (canonical theorem of judgment aggregation): For a sufficiently rich ("totally blocked") agenda the only independent and monotonic aggregation rule is the dictatorship of a particular individual.
- This observation concludes the proof of the metatheorem.

Relation to Gibbard-Satterthwaite (as a corollary)

Observation (Dietrich and List 2007): Non-manipulability is equivalent to the conjunction of independence and monotonicity (and hence to individual responsiveness in the presence of non-imposition)
-> Gibbard-Satterthwaite type results more "fundamental" for abstract aggregation theory.

Relation to Arrovian impossibility and interpretation

- (trivial) **Observation:** The Pareto property implies non-imposition.

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- (trivial) **Observation:** The Pareto property implies non-imposition.
- Is independence really the culprit for impossibility results?
  - technically yes
  - interpretationally no! because it is implied by the conjunction of two fundamental intuitions of democratic collective decision making: non-imposition and individual responsiveness.

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