Social Choice Theory - Problems, Results and Extensions

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Overview

- Main goal: Formal introduction to Social Choice Theory
- Elaborate the formal framework
- State and "prove" 3 most famous social choice results:
  - Arrow's theorem - general aspects (1951)
  - Sen's theorem - freedom aspects (1970)
  - Gibbard-Satterthwaite theorem - strategic aspects (1973/75)
- Introduce set theoretic extensions.

Historical Aspects

- "Social Choice is the study of all procedures for producing group decision from individual preferences" (McLean and Urken (1995))
  - analog to multi-criteria decision making
- First formal approaches in the 18th century
  - Marquis de Condorcet (1743-94)
  - Jean-Charles Borda (1733-99)
- In its modern form, social choice goes back to seminal work of K.J. Arrow (1951)
  - Social Choice and Individual Values
  - partly motivated as a reaction to the problems of traditional welfare economics
- Nobel Prizes: Arrow (1972), Sen (1998), Maskin (2007)

JEL classification

- According to the JEL classification it is a subfield of the analysis of Collective Decision-Making
  - JEL: D71 - Social choice; clubs; committees; associations
- According to the Mathematics Subject Classification
  - 91B14 Social choice
  - 91B12 Voting theory

Collective Decision Rule

- What are we doing when we look for a collective decision?
- Use a function (collective decision rule) that assigns to any input of individual preferences a social outcome.
  - What is the input?
  - What is the output?
  - What does the collective decision rule look like?

What is the input?

- Finite set X of alternatives/candidates or social states with certain characteristics.
- Finite set N of voters.
- Individual preferences over X by individual i are given as binary relation $R_i \subseteq X \times X$, and we write xR_iy to denote x at least as good as y in i's terms.
- Given R we can construct two related preferences P (strict preference) and I (indifference):
  - $xPy \Leftrightarrow xRy \wedge \neg yRx$
  - $xIy \Leftrightarrow xRy \wedge yRx$

Definition

A binary relation R on X is

- complete if $\forall x, y \in X$, either xRy or yRx
- reflexive if $\forall x \in X$, xRx

Preferences and Properties

Definition

A binary relation R on X is

- transitive if $\forall x, y, z \in X$, xRy and yRz implies xRz
- quasi-transitive if $\forall x, y, z \in X$, xPy and yPz implies xPz
- acyclic if $\forall x, y, z_1, ..., z_l \in X$, xPz_1, z_1Pz_2, ..., z_lPy implies xRy

Definition

R is called a *weak order* if it is complete, reflexive and transitive.

Example

Let $X = \{x, y, z\}$ and xPy,yIz and xIz. What properties does this relation satisfy?

Preference profile

Definition

A preference profile is an n-tuple of weak orders $p = (R_1, ..., R_n)$.

Usually in social choice theory we work with *linear orders*, i.e. strict rankings of the alternatives.

What is the output?

What is it that we want to get as social output?
There are various possibilities:

- singletons from X
- subsets from X
- binary relations on X
- choice functions on X

What is a choice function?

Definition (Choice function)

Let $\mathcal{X}$ be the set of all non-empty subsets of X. A choice function is a function $C : \mathcal{X} \to \mathcal{X}$ s.t. $\forall S \in \mathcal{X}$, $C(S) \subseteq S$.

Choice and preferences

Is there a relationship between choices and preferences?

Definition (Rationalizability)

A choice function C is rationalizable if there exists a preference R s.t. $\forall S \in \mathcal{X}$, $C(S) = \{x \in S : \forall y \in S, xRy\}$.

Is every choice function rationalizable by a preference R?

Example

Let $X = \{x, y, z\}$ and the choice function be s.t. $C(\{x, y, z\} = C(\{x, y\}) = y$ and $C(\{x, z\}) = C(\{y, z\}) = z$.

Collective decision rules

This somehow determines what type of collective decision rule we consider.

Definition (Preference aggregation rule)

Let $\mathcal{B}$ denote the set of all complete and reflexive binary relations on X and $\mathcal{R} \subseteq \mathcal{B}$ the set of all weak orders.
A preference aggregation rule is a mapping $f : \mathcal{R}^n \to \mathcal{B}$

Other types of collective decision rules:

- Social Welfare Function: $f : \mathcal{R}^n \to \mathcal{R}$
- Social Decision Function: $f : \mathcal{R}^n \to \mathcal{A}$, where $\mathcal{A}$ is the set of all complete, reflexive and acyclic binary relations on X.
- Social Choice Function: $f : \mathcal{R}^n \to X$

Examples of collective decision rules

- What is probably the most common rule?

Definition (Plurality Rule)

f is called plurality rule if $\forall p \in \mathcal{R}^n$ and all $x, y \in X$, $xf(p)y$ if and only if $|\{i \in N : xR_iz, \forall z \in X\}| \geq |\{i \in N : yR_iz, \forall z \in X\}|$.

Example (Plurality Rule)

1	2	2	1
a	a	b	c
b	c	c	b
c	b	a	a

- What is the plurality outcome?
  - $aPbPc$

  What happens if all completely switched their preferences?

Examples of collective decision rules

- Condorcet's idea in response to problems of plurality rule.

Definition (Simple Majority Rule)

f is called simple majority rule if $\forall p \in \mathcal{R}^n$ and all $x, y \in X$, $xf(p)y$ if and only if $|\{i \in N : xR_iy\}| \geq |\{i \in N : yR_ix\}|$.

Example (Condorcet Paradox)

R_1	R_2	R_3
a	c	b
b	a	c
c	b	a

What is the SMR-outcome? **Condorcet cycle**

Examples of collective decision rules

Definition (Borda rule)

Let $b_i(x) = |\{y \in X : xP_iy\}|$ and $b(x) = \sum_{i \in N} b_i$. Then f is called Borda rule if $\forall p \in \mathcal{R}^n$ and all $x, y \in X$, $xf(p)y$ if and only if $b(x) \geq b(y)$.

Example (Borda Rule)

R_1	R_2	R_3
a	c	d
b	b	c
c	a	b
d	d	a

What is the Borda-outcome?

Properties of social welfare functions

Definition (Unrestricted Domain)

The domain of f includes all logically possible n-tuples of individual weak orders over X.

Definition (Weak Pareto)

For all $p \in \mathcal{R}^n$ and all $x, y \in X$; $\forall i \in N$, xP_iy implies xPy.

Definition (Independence of Irrelevant Alternatives)

For all $p, p' \in \mathcal{R}^n$ and all $x, y \in X$; $\forall i \in N$, $xR_iy \Leftrightarrow xR'_iy$ implies $xRy \Leftrightarrow xR'y$.

Which social welfare functions satisfy those three conditions?
Dictatorship

Arrow's impossibility theorem

Definition (Nondictatorship)

$\nexists i \in N$ s.t. $\forall p \in \mathcal{R}^n$ and $x, y \in X$, xP_iy implies xPy.

Theorem (Arrow (1951/63))

Let $|N| \geq 2$ and $|X| \geq 3$. There exists no SWF that satisfies UD, WP, IIA and ND.

Rules and those properties

Before proving Arrow's theorem, which of the properties do certain rules violate?

- Dictatorship satisfies UD, WP, IIA but violates ND
- constant rule satisfies UD, IIA, ND but violates WP
- Borda rule satisfies UD, WP, ND but violates IIA

Example (Violation of IIA by Borda rule)

R_1	R_2	R_3	R'_1	R'_2	R'_3
a	c	d	b	c	d
b	b	c	a	b	c
c	a	b	d	a	b
d	d	a	c	d	a

Proof of Arrow's theorem

For the proof we need the following definitions:

Definition (Decisiveness)

$G \subseteq N$ is decisive over the ordered pair $\{x, y\}$, $\bar{D}_G(x, y)$ iff xP_iy, $\forall i \in G$ implies xPy.

Definition (Almost decisiveness)

$G \subseteq N$ is almost decisive over ordered pair $\{x, y\}$, $D_G(x, y)$ iff xP_iy, $\forall i \in G$ and yP_ix, $\forall i \in N \backslash G$ implies xPy.

Two lemmata (Sen)

The proof of Arrow's theorem is achieved in different forms. One is via the following two lemmata:

Lemma (Field expansion lemma)

For any SWF satisfying UD, WP and IIA and $|X| \geq 3$, *if a group* G *is almost decisive over some ordered pair* $\{x, y\}$, *then it is decisive over every ordered pair, i.e.*
$[\exists x, y \in X : D_G(x, y)] \Rightarrow \left[\forall a, b \in X : \bar{D}_G(a, b)\right]$

Lemma (Group contraction lemma)

For any SWF satisfying UD, WP and IIA and $|X| \geq 3$, *if any group* G *with* $|G| > 1$ *is decisive, then so is some proper subgroup of* G.

Field expansion lemma

Consider $X = \{x, y, a, b\}$ and the following profile:

$i \in G$	$k \in N \backslash G$
a	$aP_k x$
x	$yP_k b$
y	$yP_k x$
b	

- aPx and yPb because of WP
- xPy because of $D_G(x, y)$
- aPb because of (quasi) transitivity of f
- by IIA this only depends on orderings of a and b of which only those in group G have been specified
- Hence: $\bar{D}_G(a, b)$
  - and hence no difference between being almost decisive over some ordered pair and being decisive over every ordered pair.

Group contraction lemma

Partition G into G_1 and G_2

G_1	G_2	$k \in N \setminus G$
x	y	z
y	z	x
z	x	y

- yPz by decisiveness of G
  - 3 possibilities: $xPyPz$, $yPxPz$, $yPzPx$
  - $yPxPz$ cannot occur as both G_1 (over $\{x,z\}$) and G_2 (over $\{y,x\}$) would be almost decisive.
- If $xPyPz$, then G_1 almost decisive over $\{x,z\}$
- If $yPzPx$, then G_2 almost decisive over $\{y,x\}$
- from field expansion lemma either G_1 or G_2 is decisive

Proof of Arrow's theorem

Proof.

- WP and field expansion lemma implies that N is decisive
- by the group contraction lemma we can eliminate members of N until we are left with a dictator.

$\square$

Proofs and resolutions

- Other proof techniques have been used
  - using a geometric approach
  - using ultrafilters
- Ways to overcome the negative results?
  - Relaxing the consistency conditions of the social outcome to quasi-transitivity or acyclicity.
  - Use of broader informational basis, i.e. interpersonal comparisons
  - but this often leads to other "dictator-like" results with veto rights or oligarchies
- another approach: domain restrictions
  - single-peaked preferences

Single-peaked preferences

- Sometimes there are natural restrictions on the domain
  - preferences based on distance from an optimal point on a line
  - left-right orientation in politics

R_1	R_2	R_3	R'_1	R'_2	R'_3
a	c	b	a	c	c
b	a	c	c	a	b
c	b	a	b	b	a

- Single-peaked preferences are sufficient for a Condorcet-winner to occur!

Sen's Liberal Paradox

We have not considered any aspects of choices among alternatives that lie in one's private domain.

> *[Sen, 1970] If you prefer to have pink walls rather than white, the society should permit you to have this even if a majoritiy of the community would like to see your walls white.*

Sen's liberal paradox

Let $f : \mathcal{R}^n \to \mathcal{A}$ be a social decision function and consider the following property:

Definition (Minimal Liberalism)

There exist at least 2 individuals s.t. each of them is decisive over at least one pair of alternatives, i.e. if i is decisive over (x, y), then $xP_iy \Rightarrow xPy$.

Theorem (Sen (1970))

There exists no social decision function satisfying UD, WP and ML.

Proof

Proof.

Let $X = \{x, y, z\}$ and $i, j \in N$ be such that $\bar{D}_i(x, y)$ and $\bar{D}_j(z, x)$. The preferences are considered as follows:

R_i	R_j	$k \neq i, j$
x	y	yP_kz
y	z	
z	x	

- xPy because of ML of i
- yPz because of WP
- zPx because of ML of j
- Leads to a cycle!

$\square$

Relevance

- liberal values conflict with the Pareto principle in a basic sense
- Compared to Arrow's theorem
  - it also works if we just consider the possibility of choices, i.e. acyclic social preferences
  - it does not use the rather criticized IIA condition
  - there is no satisfactory resolution via a broadening of the informational basis through interpersonal comparisons

Strategic aspects in voting

Strategic aspects in voting have been known for a long time:

> *My scheme is only intended for honest men! [Borda]*
>
> *Voters adopt a principle of voting which makes it more of a game of skill than a real test of the wishes of the electors. [Dodgson]*
>
> *Politicians are continually poking and pushing the world to get the results they want. The reason they do this is they believe (and rightly so) that they can change outcomes by their efforts. It is often the case that voting need not have turned out the way it did. [Riker]*

Manipulability

Let $p = (R_1, ..., R_n) \in \mathcal{R}^n$ and let (p_{-i}, p'_i) denote the profile $p' = (R_1, ..., R_i^{'}, ..., R_n)$. Now:

Definition (Manipulability)

Social choice function $f : \mathcal{R}^n \to X$ is manipulable by i at profile p via $R_i^{'}$ if $f(p')P_i f(p)$.

Theorem (Gibbard-Satterthwaite (1973/75))

Let $|N| \geq 2$ and $|X| \geq 3$. If f is non-manipulable and satisfies WP, it is a dictatorship.

Manipulability

	xyz	xzy	yxz	yzx	zxy	zyx
xyz						
xzy						
yxz						
yzx						
zxy						
zyx						

Manipulability

	xyz	xzy	yxz	yzx	zxy	zyx
xyz	x	x				
xzy	x	x				
yxz			y	y		
yzx			y	y		
zxy					z	z
zyx					z	z

- If both top-rank an item, it has to be chosen!
- Can we say something else using the WP-criterion?
  - there are some items that cannot be chosen at certain profiles
  - what is possible at (xzy, yxz)?
  - only x or y

Manipulability

	xyz	xzy	yxz	yzx	zxy	zyx
xyz	x	x	x, y	x, y	x, z	
xzy	x	x	x, y		x, z	x, z
yxz	x, y	x, y	y	y		y, z
yzx	x, y		y	y	y, z	y, z
zxy	x, z	x, z		y, z	z	z
zyx		x, z	y, z	y, z	z	z

- Choice function requires a single outcome

Manipulability

	xyz	xzy	yxz	yzx	zxy	zyx
xyz	x	x	x, y	x, y	$\mathbf{z}$	
xzy	x	x		x, y	x, z	x, z
yxz	x, y	x, y	y	y		y, z
yzx	x, y		y	y	y, z	y, z
zxy	x, z	x, z		y, z	z	z
zyx		x, z	y, z	y, z	z	z

- Assume z chosen at profile (xyz, zxy).
- What does strategy-proofness imply?
  - z needs to be the outcome for the whole column. Why?
  - Player 1 could manipulate at profile (xyz, zxy) by reporting a preference that gives either x or y as outcome.

Manipulability

	xyz	xzy	yxz	yzx	zxy	zyx
xyz	x	x	x, y	$\mathbf{x}, \mathbf{y}$	z	
xzy	x	x		x, y	z	x, z
yxz	x, y	x, y	y	y	z	y, z
yzx	x, y		y	y	z	y, z
zxy	x, z	x, z		y, z	z	z
zyx		x, z	y, z	y, z	z	z

- What can be the outcome at (xzy, yzx)?
- Only y as otherwise player 2 could manipulate by enforcing outcome z.
- Now the whole column needs to choose y. Otherwise player 1 can manipulate.
- can be continued until the table looks as follows:

Manipulability

	xyz	xzy	yxz	yzx	zxy	zyx
xyz	x	x	y	y	z	z
xzy	x	x	y	y	z	z
yxz	x	x	y	y	z	z
yzx	x	x	y	y	z	z
zxy	x	x	y	y	z	z
zyx	x	x	y	y	z	z

- But this makes player 2 a dictator!

Other important properties

- Another important property is *monotonicity*.

Definition (Monotonicity)

If x is winning with f at profile p and p' is determined from p by increased support for x (i.e. moving up x in the individuals rankings without changing anything else), then x has to be winning with f at profile p'.

Example (Plurality Runoff Rule)

34	35	31	30	4	35	31
a	b	c	a	b	b	c
c	c	b	c	a	c	b
b	a	a	b	c	a	a

Set-Extensions

- How should we rank sets of objects x and y?
- How is this related to a possibly underlying ranking (xPy) of the basic objects?
- Assume 3 sets $\{x, y\}$ vs. $\{x\}$ vs. $\{y\}$
- 6 possible rankings
- ranking would be based on different underlying reasonings
- New notation: Let $\mathcal{X}$ be the set of all subsets of X and $\succsim \subseteq \mathcal{X} \times \mathcal{X}$ be a relation on $\mathcal{X}$.

Set-Extensions

$$\{x\} \succ \{x, y\} \succ \{y\}$$

- x and y are mutually exclusive - only one of them occurs
- $\{x, y\}$ could be considered as lottery
- agent as expected utility maximizer

$$\{x\} \succ \{x, y\} \sim \{y\}$$

- sets of possible outcome ranked on the basis of the worst-case scenario
- maximin criterion

Set-Extensions

$$\{x\} \sim \{x, y\} \succ \{y\}$$

- as before - agent ranks according to optimistic evaluation
- maximax criterion

$$\{x, y\} \succ \{x\} \succ \{y\}$$

- agent has the final choice from $\{x, y\}$
- not only concerned with the alternative she receives, but also the freedom of choice
- if alternatives are compatible and desirable, having both of them might be preferred

Set-Extensions

$$\{x\} \succ \{y\} \succ \{x, y\}$$

- as before - but alternatives might be undesirable

$$\{x, y\} \succ \{x\} \sim \{y\}$$

- agent purely interested in the freedom of choice

Set-Extensions

What are fields where we need to be explicit about ranking sets of objects?

- Voting theory
  - cases of multiple outcomes
  - or as a modelling tool for choice under complete uncertainty (agent knows the set of possible consequences of an action but cannot assign probabilities to those outcomes)
- matchings and assignments
  - allocation of (sets of) workers to firms, or students to colleges, etc.
  - major interest in stability issues and strategic behaviour
- fundamental issues in ethics and economics
  - examination of the instrumental and the intrinsic value of freedom of choice
  - formalizing notions such as equality of opportunity

Axioms

- What are reasonable axioms?

Definition (Dominance)

$\forall A \in \mathcal{X}, \forall x \in X$, $[xPy, \forall y \in A] \Rightarrow A \cup \{x\} \succ A$ and $[yPx, \forall y \in A] \Rightarrow A \succ A \cup \{x\}$

Definition (Independence)

$\forall A, B \in \mathcal{X}, \forall x \in X \backslash (A \cup B), A \succ B \Rightarrow A \cup \{x\} \succsim B \cup \{x\}$

Theorem (Kannai and Peleg (1984))

Suppose R is a linear ordering on X and $\succsim$ is a reflexive and QT relation on $\mathcal{X}$. If $\succsim$ satisfies dominance and independence, then $A \sim \{max(A), min(A)\}$ for all $A \in \mathcal{X}$

Axioms

Definition (Simple Dominance)

$\forall x, y \in X, xPy \Rightarrow [\{x\} \succ \{x, y\} \& \{x, y\} \succ \{y\}]$

Definition (Simple Uncertainty Aversion)

For all $x, y, z \in X$, $xPyPz \Rightarrow \{y\} \succ \{x, z\}$

Definition (Simple Top Monotonicity)

For all $x, y, z \in X$, $xPyPz \Rightarrow \{x, z\} \succ \{y, z\}$

Theorem (Bossert, Pattanaik and Xu (2000))

$\succsim$ satisfies simple dominance, independence, simple uncertainty aversion and simple top monotonicity if and only if $\forall A, B \in \mathcal{X}$, $A \succsim B \Leftrightarrow (\min(A) P \min(B)$ or $[\min(A) = \min(B)$ and $\max(A) R \max(B)])$.

Axioms

We can extend this to get to lexicographic rules.

Definition (Top Independence)

$\forall A, B \in \mathcal{X}$, $\forall x \in X$ such that xPy, $\forall y \in A \cup B$,
$A \succ B \Rightarrow A \cup \{x\} \succ B \cup \{x\}$

Definition (Disjoint Independence)

$\forall A, B \in \mathcal{X}$ such that $A \cap B = \emptyset$, $\forall x \in X \backslash (A \cup B)$,
$A \succ B \Leftrightarrow A \cup \{x\} \succ B \cup \{x\}$

Theorem (Pattanaik and Peleg (1984))

Suppose $|X| \geq 4$, R is a linear ordering on X and $\succsim$ is a quasi-ordering on $\mathcal{X}$. $\succsim$ satisfies dominance, neutrality, top independence and disjoint independence if and only if $\succsim$ is the leximax ordering.

Axioms

Example (Leximax-Ordering)

Let $X = \{a, b, c, d, e\}$ with $aPbPcPdPe$. How are the sets $A = \{a, c, d, e\}$, $B = \{b, c, d\}$ and $C = \{a, d\}$ related to each other according to the leximax-ordering of $\mathcal{X}$?

- $A \succ B$
- $A \succ C$
- $C \succ B$

Axioms

Can also use this to analyse opportunities - freedom of choice

Definition (Indifference Between No-Choice Situations)

$\forall x, y \in X$, $\{x\} \sim \{y\}$

Definition (Simple Expansion Monotonicity)

For all distinct $x, y \in X$, $\{x, y\} \succ \{x\}$

Definition (Strong Independence)

$\forall A, B \in \mathcal{X}$, $\forall x \in X \backslash (A \cup B)$, $A \succsim B \Leftrightarrow A \cup \{x\} \succsim B \cup \{x\}$

Theorem (Pattanaik and Xu (1990))

Suppose $\succsim$ is a transitive relation on $\mathcal{X}$. $\succsim$ satisfies indifference between no-choice situations, simple expansion monotonicity, and strong independence if and only if $A \succsim B \Leftrightarrow |A| \geq |B|$

Conclusion

- We have discussed 3 of the major impossibility results in Social Choice Theory
  - There is an inconsistency between basic reasonable properties. [Arrow]
  - There is an inconsistency between basic liberal aspects and the Pareto principle. [Sen]
  - There is an inconsistency between basic strategic aspects and the Pareto principle. [Gibbard-Satterthwaite]
- Extensions from rankings of objects to ranking sets of objects

Some Basic Literature

- Arrow, K.J. (1963): Social choice and Individual Values (2nd ed.). Yale University Press, New Haven.
- Austen-Smith, D. and J.S. Banks (2000): Positive Political Theory I. University of Michigan Press, Ann Arbor.
- Barbera, S., Bossert, W. and P.K. Pattanaik (2004): Ranking Sets of Objects. In: Barbera, S., Hammond, P. and C. Seidl. Handbook of Utility Theory. Springer, Berlin.
- Bossert, W., Pattanaik, P.K. and Y. Xu (2000): Choice under complete uncertainty: axiomatic characterizations of some decision rules. *Economic Theory*, **16**, 295–312.
- Gibbard, A. (1973): Manipulation of voting schemes: a general result. *Econometrica*, **41**, 587–601.
- McLean, I. and A.B. Urken (eds.) (1995): Classics of Social Choice. University of Michigan Press, Ann Arbor.

Some Basic Literature

- Riker, W.H. (1982): Liberalism against populism. W.H. Freeman and Company.
- Satterthwaite, M. (1975): Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, **10**, 187–217.
- Sen, A.K. (1970): Collective Choice and Social Welfare. North Holland, Amsterdam.
- Sen, A.K. (1970): Impossibility of a Paretian Liberal. *Journal of Political Economy*, **78**, 152–157.
- Sen, A.K. (1986): Social choice theory. in: Arrow, K.J. and M.D. Intriligator, Handbook of Mathematical Economics. Elsevier.