Distance-based approaches to aggregation: approaching consensus?

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- Well known problems of aggregation rules: Classical impossibility results of Arrow ("every otherwise satisfactory aggregation rule is the dictatorship of a particular individual") and Gibbard-Satterthwaite ("every non-dictatorial aggregation rule is manipulable")
- Negative results largely recovered by the recent extensions of Arrovian social choice theory (abstract aggregation theory, judgment aggregation, computational social choice)

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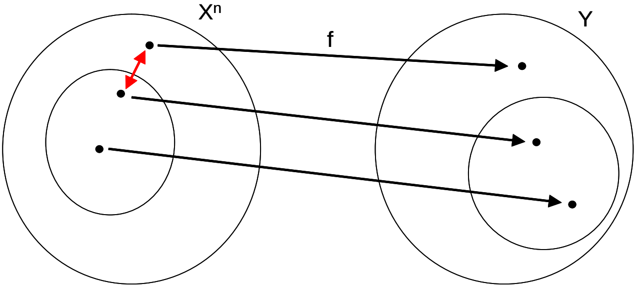
- For subsets of the profiles in the domain of an aggregation rule the assignment of a collective outcome is uncontroversial (e.g. for unanimous profiles).
- Can this consensus be extended to the whole domain by assigning to any profile the outcome of the consensual profile which is closest to it?
- diagrammatically:



Distances in the social choice literature

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- (Characterization) Results, e.g. Borda winner characterized by its closeness to being a unanimous winner
  (for a survey see Meskanen and Nurmi 2008)
- Related: geometric representation and analysis of abstract aggregation problems (Eckert/Klamler 2009; Pivato 2009)

Kemeny: the archetypical distance

Definition

Let $R_1, R_2 \in \mathcal{B}$, the function $d : \mathcal{B} \times \mathcal{B} \longrightarrow \Re$ is called *distance function* on $\mathcal{B}$.

Thus, for any two preferences R_1 and R_2, $d(R_1, R_2)$ represents the distance between R_1 and R_2.

Axiom

1. $d(R_1, R_2) \geq 0$ *(non-negativity)*
2. $d(R_1, R_2) = 0$ *if and only if* $R_1 = R_2$
3. $d(R_1, R_2) = d(R_2, R_1)$ *(symmetry)*
4. $d(R_1, R_2) + d(R_2, R_3) \geq d(R_1, R_3)$ *(triangle inequality)*

Specific properties of the Kemeny distance

Neutrality among alternatives.

Axiom

If R_1' is derived from R_1 and R_2' from R_2 via a permutation of X, then $d(R_1, R_2) = d(R_1', R_2')$.

Distance can only depend on parts of the preferences which are different.

Axiom

Let $X = S \cup T$ with $S \cap T = \emptyset$ and aR_ib for all $a \in S$, $b \in T$ and $i \in \{1, 2\}$. If R_1 and R_2 fully agree upon S, written $R_1|S = R_2|S$, then $d(R_1, R_2) = d(R_1|T, R_2|T)$. The analogous needs to hold if $R_1|T = R_2|T$.

Unit of measurement.

Axiom

If $R_1 \neq R_2$, then $d(R_1, R_2) \geq 1$, i.e. the minimal positive distance is 1.

Kemeny

Definition

A distance function $d : \mathcal{R} \times \mathcal{R} \rightarrow \Re_+$ is called *Kemeny distance* if for all R_1, R_2, $d(R_1, R_2) = |(R_1 \backslash R_2) \cup (R_2 \backslash R_1)|$.

This distance function d_K is hence the cardinality of the symmetric difference between R_1 and R_2. Differently speaking, it counts (twice) the number of inversions of pairs in the respective preferences R_1 and R_2.

Example

Example

Let $X = \{a, b, c\}$.

R_1	R_2	R_3
a	b	c
b	a	b
c	c	a

Table: Rankings for 3 alternatives.

Preference R_1 differs from preference R_2 by a difference in the relative ranking between alternatives a and b, i.e. a full inversion between those two alternatives leads from one preference to the other. Hence, the symmetric difference between R_1 and R_2 is $(R_1 \backslash R_2) \cup (R_2 \backslash R_1) = \{(a, b), (b, a)\}$, leading to a Kemeny distance of $d_K(R_1, R_2) = 2$.

Kemeny axiomatically

The Kemeny distance is not only an intuitively plausible way to measure distances between preferences: Interestingly, the Kemeny distance d_K is also the only distance function to satisfy the above axioms.

Theorem

(Kemeny 1959) Distance function d is the Kemeny distance d_K if and only if it satisfies axioms 1-4

Kemeny and aggregation

Probably the most common way to aggregate individual preferences is via simple majorities. For any profile $p \in \mathcal{R}^n$ and alternatives $a, b \in X$, let $m_{ab}(p) = |\{i \in N : aR_i b\}| - |\{i \in N : bR_i a\}|$, i.e. the majority margin between a and b.

Definition

The aggregation rule $m : \mathcal{R}^n \to \mathcal{B}$ is the *simple majority rule* if for all $p \in \mathcal{R}^n$, $a \; m(p) \; b$ if and only if $m_{ab} \geq 0$.

Kemeny aggregated and considered as an aggregation rule

Definition (Kemeny Rule)

Given a profile $p = (R_1, ..., R_n)$ and a preference R, let $\underline{\mathrm{d}}(p, R) = \sum_{i=1}^{n} d(R_i, R)$ measure the distance between a profile p and a preference R. An aggregation rule $f : \mathcal{R}^n \rightarrow \mathcal{R}$ is the *Kemeny rule* f_K if and only if for all $p \in \mathcal{R}^n$,
$f(p) = \{R \in \mathcal{R} : \forall R' \in \mathcal{R}, \underline{\mathrm{d}}(p, R) \leq \underline{\mathrm{d}}(p, R')\}$.

Example

Let $X = \{a, b, c\}$, $|N| = 5$

2	2	1
a	c	b
b	a	c
c	b	a

Table: Rankings for 3 alternatives.

a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a
12	14	18	16	12	18

Table: Distances from profile in Table 2.

Desirable properties of the Kemeny rule

- The Kemeny rule is a so-called *Condorcet extension* (see Fishburn 1977), as its outcome is identical to majority rule as long as there are no preference cycles. Actually, the profile in the previous example leads to a preference cycle as a is socially preferred to b, b to c and c to a.
- As previously for the Kemeny distance, also for the Kemeny rule there exists a strong axiomatic foundation established by Young and Levenglick (1978).

Consistency

Axiom

Let $V \subseteq N$ and (S, T) be a partition of V, i.e. S and T are disjoint "electorates". Denote a profile p restricted to the set $W \subseteq N$ of individuals by $p|W$. An aggregation rule f is consistent if and only if for all $p \in \mathcal{R}^n$, $f(p|S) \cap f(p|T) \neq \emptyset$ implies $f(p|V) = f(p|S) \cap f(p|T)$.

Consistency requires that any (set of) preference(s) considered best in two disjoint electorates needs to be considered best also when those electorates decide jointly.

Condorcet property

Axiom

For any profile $p \in \mathcal{R}^n$, *and aggregation rule* f *let* $a\ \hat{f}(p)\ b$ *denote the fact that* $a\ f(p)\ b$ *and for all* $c \in X$, $c\ f(p)\ b \rightarrow c\ f(p)\ a$ *and* $a\ f(p)\ c \rightarrow b\ f(p)\ c$, *i.e.* a *and* b *are neighbors in the preference ranking determined by* f. *An aggregation rule* f *is called Condorcet if* $m_{ab} > 0$ *implies not* $b\ \hat{f}(p)\ a$ *and* $m_{ab} = 0$ *implies* $a\ \hat{f}(p)\ b$ *if and only if* $b\ \hat{f}(p)\ a$.

The Condorcet property guarantees that whenever there is a majority preferring alternative *a* over *b*, then it cannot be the case that the aggregation rule ranks *b* immediately above *a*. This has some intuitive appeal in the sense that otherwise a simple switch of those two alternatives - without affecting any other alternative or pair of alternatives - would benefit a majority of the electorate.

Neutrality

Axiom

Let π be a permutation of X and denote by $\pi(p)$ the permuted profile p. An aggregation rule f is said to be neutral if for all $p \in \mathcal{R}^n$, $a\ f(p)\ b$ if and only if $\pi(a)\ f(\pi(p))\ \pi(b)$.

Neutrality ensures that alternatives are treated equally, i.e. aggregation results do not depend on the names of the alternatives.

Theorem

(Young and Levenglick 1978) The Kemeny rule is the only aggregation rule that satisfies the axioms neutrality, Condorcet and consistency.

Other distance-based aggregation rules

Definition (Slater rule)

An aggregation rule $f : \mathcal{R}^n \to \mathcal{R}$ is the *Slater rule* f_S if and only if for all $p \in \mathcal{R}^n$, $f(p) = \{R \in \mathcal{R} : \forall R' \in \mathcal{R}, d_K(m(p), R) \leq d_K(m(p), R')$.

Definition (Dodgson rule)

Let $\mathcal{R}^n_x$ be the set of profiles in which $x \in X$ is a strict Condorcet winner and let $D : \mathcal{R}^n \times \mathcal{R}^n \to \Re_+$ be a distance function on the set of profiles such that for all $p, p' \in \mathcal{R}^n$, $D(p, p') = \sum_{i \in N} d_K(R_i, R_i')$. An aggregation rule $f : \mathcal{R}^n \to \mathcal{R}$ is the *Dodgson rule* f_D if and only if for all $p \in \mathcal{R}^n$, and all $a, b \in X$, $a\ f(p)\ b$ if and only if $\min_{p' \in \mathcal{R}^n_a} D(p, p') \leq \min_{p' \in \mathcal{R}^n_b} D(p, p')$.

"Distances" (dissimilarities) between distance-based aggregation rules

- The top ranked alternative according to the Kemeny rule can be bottom ranked according to the Dodgson rule and vice versa.
- The top ranked alternative according to the Dodgson rule can be bottom ranked by any scoring rule.
- The top ranked alternative by the Slater rule can be bottom ranked by the Dodgson rule.
- and many others (for a survey see Eckert/Klamler 2011).

Problems with consistency conditions for distance-based approaches in general

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- Two different types of consistency problems can emerge with distance-based consistency conditions:
- Impossibility results, e.g. Baigent's (1987b) impossibility of proximity preservation
- Anything goes results (Elkind/Faliszweski/Slinko 2010): (Almost) any aggregation rule can be metrically rationalized by some distance.

Abstract framework

Ω: set of **possible worlds** (interpretation: set of all possible complete descriptions of the state of the world by an individual, e.g. linear preference orderings)

$\Omega/\equiv$: partition of Ω according to the equivalence relation $\equiv\subset\Omega\times\Omega$ (interpretation: partition of Ω into **equivalence classes** corresponding to the aspects which are relevant for the collective decision, e.g. top rank of an alternative)

Observe that this partition is allowed to be the finest possible (e.g. in the case of a social welfare function)!

$$\boxed{\begin{array}{cc} x & x \\ y & z \\ z & y \end{array}}\ \boxed{\begin{array}{cc} y & y \\ x & z \\ z & x \end{array}}\ \boxed{\begin{array}{cc} z & z \\ x & y \\ y & x \end{array}} \qquad \boxed{\begin{array}{c} x \\ y \\ z \end{array}}\ \boxed{\begin{array}{c} x \\ z \\ y \end{array}}\ \boxed{\begin{array}{c} y \\ x \\ z \end{array}}\ \boxed{\begin{array}{c} y \\ z \\ x \end{array}}\ \boxed{\begin{array}{c} z \\ x \\ y \end{array}}\ \boxed{\begin{array}{c} z \\ y \\ x \end{array}}$$

"Top rank" partition   Finest partition

Formal framework ctd.

$\mathcal{U} = \left\{ \bigcup_{S \in P} S | P \in \mathcal{P}(\Omega/ \equiv) \backslash \varnothing \right\}$: set of all **unions of equivalence classes** (interpretation: the larger the union the less valuable is the information for the collective decision)

N: set of **individuals**

Ω^N (denoted by Π): set of **profiles** $\pi = (\pi_1, \pi_2, ..., \pi_{|N|})$ of possible worlds (interpretation: lists of individual descriptions of the state of the world)

Definition

An **aggregation rule** is a mapping $F : \Pi \rightarrow \mathcal{U}$ which assigns to each profile $\pi \in \Pi$ an equivalence class or a union of equivalence classes $F(\pi) \in \mathcal{U}$.

Consensus

Definition

A **consensus class** is a pair $\mathcal{C} = (C, f)$ where $C \subseteq \Pi$ is the set of **consensual profiles** and the **consensus mapping** $f : C \rightarrow \Omega/\equiv$ is an onto mapping which assigns to every consensual profile $\pi \in C$ an equivalence class in $\Omega/\equiv$.

Example

Unanimity is the consensus class $\mathcal{C} = (C, f)$ where $C = \{\pi \in \Pi | (\forall i, j \in N)\pi_i = \pi_j\}$ is the set of unanimous profiles and $f : C \rightarrow \Omega/\equiv$ assigns to every unanimous profile of possible worlds the equivalence class of this possible world.

Definition

An aggregation rule $F : \Pi \rightarrow \mathcal{U}$ is **consistent with a consensus class** $\mathcal{C}_F = (C, f)$ if for all profiles $\pi \in C$, $F(\pi) = f(\pi)$.

Distance rationalizability

Definition

An aggregation rule $F : \Pi \to \mathcal{U}$ is **distance rationalizable** via a consensus class $\mathcal{C}_F = (C, f)$ and a distance $d_F : \Pi \times \Pi \to R_+$ (is $(\mathcal{C}_F, d_F)$-rationalizable) if for all equivalence classes $[\omega] \in \Omega/\equiv$ and for any profile $\pi \in \Pi$

$$F(\pi) = \bigcup \left\{ [\omega] \in \Omega/\equiv \mid [\omega] = f\left(\min_{\pi' \in C} d(\pi, \pi') \right) \right\},$$

i.e. if the outcome is the union of all equivalence classes associated by the consensus mapping f with the distance minimizing consensual profile(s).

Anything goes...

Theorem

For any aggregation rule $F : \Pi \rightarrow \mathcal{U}$ *which is consistent with a consensus class* $\mathcal{C}_F = (C, f)$ *there exists a distance* $d_F : \Pi \times \Pi \rightarrow \mathrm{R}_+$ *such that* F *is* $(\mathcal{C}_F, d_F)$*-rationalizable.*

... with the right distance

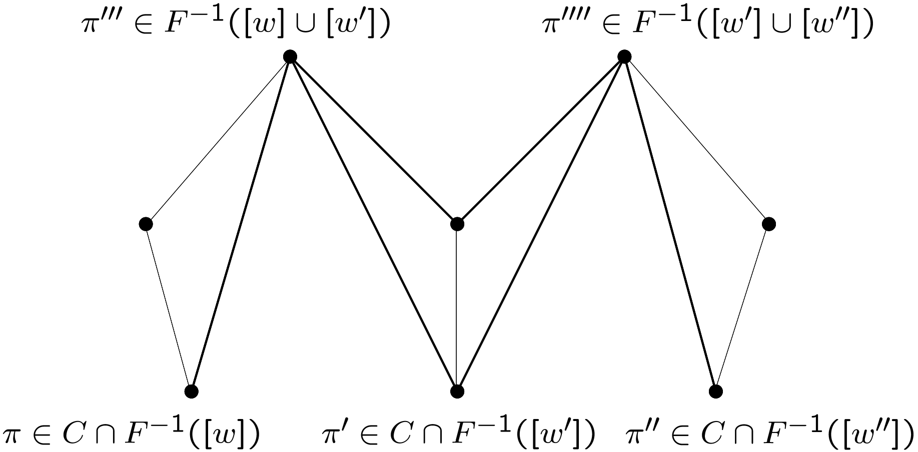
Lemma

For any aggregation rule $F : \Pi \to \mathcal{U}$ which is consistent with a consensus class $\mathcal{C}_F = (C, f)$ the undirected graph $G = (\Pi, E)$ defined, for any distinct $\pi, \pi' \in \Pi$ by

$$\{\pi, \pi'\} \in E \text{ whenever } F(\pi) \in \Omega/\equiv \text{ and } F(\pi) \subseteq F(\pi')$$

induces a shortest path distance $d_F : \Pi \times \Pi \to R_+$.

(Neighborhood structure of) the graph



Anything goes (proof)

Proof.

First, assume that $\pi \in C$. Then $\{\pi' \in \Pi | d_F(\pi, \pi') = 0\} = \pi$ (by the identity of indiscernables) and $F(\pi) = f(\pi) \in \Omega/\equiv$ (by consistency). If $\pi \notin C$, then $d_F(\pi, \pi') \geq 1$ for any profile $\pi' \in \Pi$, and for any profile $\pi' \in C$ we have $d_F(\pi, \pi') = 1$ if $F(\pi') \subseteq F(\pi)$. Morevover, for any equivalence class $[\omega] \nsubseteq F(\pi)$ and any profile $\pi' \in C$ such that $f(\pi') = [\omega]$ we have $d_F(\pi, \pi') \geq 2$.

Thus $F(\pi) = \bigcup \left\{ [\omega] \in \Omega/\equiv \,|\, [\omega] = f\left(\min_{\pi' \in C} d(\pi, \pi') \right) \right\}$. □

Distance-based consistency: too weak or too strong

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Distance-based consistency: too weak or too strong

- Anything goes results driven by non-neutrality of the distance with respect to the consensus class (Formally: the graph from which the distance is derived is given by the neighborhoods of the consensual profiles)
- But is it reasonable to require consistency with respect to an "objective" distance?
- Baigent type results about the impossibility of proximity preservation can be understood as a negative answer to this question.

Proximity preservation

Definition

Let $d : \Pi \times \Pi \to R_+$ and $\delta : \mathcal{U} \times \mathcal{U} \to R_+$ be a distance on the domain, respectively on the codomain, of the aggregation rule $F : \Pi \to \mathcal{U}$.
F satisfies **proximity preservation** if for any profiles $\pi, \pi', \pi'' \in \Pi$

$$d(\pi, \pi') < d(\pi, \pi'') \Rightarrow \delta(F(\pi), F(\pi')) \leq \delta(F(\pi), F(\pi'')).$$

Unfortunately, the imposition of this property makes any reasonable aggregation rule incompatible with any reasonable neutral distance on profiles!

Reasonable aggregation rules and distances

Definition

An aggregation rule $F : \Pi \to \mathcal{U}$ satisfies **minimal compensation** if it is not the case that for all pairs of i-variants $\pi = (\pi_1, \pi_2, .., \pi_i, ..., \pi_{|N|})$ and $\pi' = (\pi_1, \pi_2, ..., \pi'_i, ..., \pi_{|N|})$ such that $F(\pi) \neq F(\pi')$ there does not exist a profile $\pi'' = (\pi''_1, \pi''_2, ..., \pi'_i, ..., \pi''_{|N|})$ such that $F(\pi'') = F(\pi)$, i.e. there does not exist a change from π' to π'' that compensates the change from π to π' while keeping the pivotal characteristic π'_i.

Weaker condition than anonymity!

Definition

A distance $d : \Pi \times \Pi \to R_+$ over profiles is **monotonic** if for all profiles $\pi, \pi', \pi'' \in \Pi$
$d(\pi, \pi') < d(\pi, \pi'')$ whenever π'' differs from π in more components than π'.

Natural condition!

The impossibility of proximity preservation

Theorem

There does not exist an aggregation rule $F : \Pi \to \mathcal{U}$ which satisfies minimal compensation and proximity preservation with respect to a monotonic distance on profiles.

Proof.

By **minimal compensation** there exists a pair of i-variants $\pi = (\pi_1, \pi_2, .., \pi_i, ..., \pi_{|N|})$ and $\pi' = (\pi_1, \pi_2, ..., \pi'_i, ..., \pi_{|N|})$ such that $F(\pi) \neq F(\pi')$ and a profile $\pi'' = (\pi''_1, \pi''_2, ..., \pi'_i, ..., \pi''_{|N|})$ such that $F(\pi'') = F(\pi)$. Hence, by monotonicity, $d(\pi, \pi') < d(\pi, \pi'')$, but, by construction, $\delta(F(\pi), F(\pi'')) = 0 < \delta(F(\pi), F(\pi'))$, which violates proximity preservation. □

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Conclusion

- Distance-based approaches are useful for the characterization of existing aggregation rules and eventually for the design of new ones.
- But metric rationalization in general does not guarantee more "rationality" in aggregation as long as the corresponding consistency conditions can be seen as either too weak or too strong.
- Possible escape route: "Rationalizing Distance Rationalizability" via dynamic social choice with a reward driven process of preference adaptation (Boutilier/Procaccia 2012)

References

Baigent, N. (1987a): "Metric rationalisation of social choice functions according to principles of social choice", Mathematical Social Sciences 13: 59-65

Baigent, N. (1987b): "Preference proximity and anonymous social choice", Quarterly Journal of Economics 102: 161-169.

Boutilier, C. and and A. D. Procaccia (2012): "A Dynamic Rationalization of Distance Rationalizability", AAAI-12: Proc. 26th AAAI Conference on Artificial Intelligence, Jul 2012 (forthcoming).

Eckert, D. and Ch. Klamler (2009): "A Geometric Approach to Paradoxes of Majority Voting: From Anscombe's Paradox to the Discursive Dilemma with Saari and Nurmi", Homo Oeconomicus, 26, 3/4, 471-488.

Eckert, D. and Ch. Klamler (2011): "Distance-Based Aggregation Theory", in: E. Herrera-Viedma et al, eds, Consensual Processes. Studies in Fuzziness and Soft Computing vol. 267, 3-22, Springer.

Elkind, E., P. Faliszewski and A. Slinko (2010): On the role of distances in defining voting rules, Proc. 9th Int Conf. om Autonomous Agents and Multiagent Systems.

Fishburn, PC (1977): "Condorcet Social Choice Functions" SIAM J Appl Math, 33, 469-489.
Kemeny, K. (1959) "Mathematics without numbers", Daedalus 88 577–591.
Lehrer, E. and S. Nitzan (1985): "Some General Results on the Metric Rationalization for Social Decision Rules", Journal of Economic Theory, 37(1), 191-201.
Meskanen, T. and H. Nurmi (2008): Closeness counts in social choice, in: Braham, M. and F. Steffen, eds, Power, Freedom and Voting, Springer, Berlin.
Pivato, M. (2009) "Geometric models of consistent judgement aggregation," Social Choice and Welfare, 33(4), 559-574
Young, H.P. and A. Levenglick (1978): "A Consistent Extension of Condorcet's Election Principle", SIAM J Appl Math 35, 285–300.